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Management Production Functions and the Theory of the Firm

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1. INTRODUCTION

Management is often said to be the one factor that limits the size of firms. Now management is itself subject to the economies of the division of labor. Through organization the indivisibility of the single owner–manager may be overcome. If it is believed that increasing size of the organization burdens every productive worker with an ever increasing number of administrators per production worker, then this claim is shown to be wrong both on theoretical [1] and empirical grounds [6]. In fact, looking at business organizations in this way, the opposite result would appear to be true: There is no optimal size of the firm; almost any size is equally efficient in terms of administrative overhead per worker.

There is, however, one piece missing: How is the productivity of workers affected by the size of the organization? Is there possibly a systematic relationship between the span of control and organization size [6]? In any case, how is the optimal span of control determined? Why should it be constant?

To answer these and related questions, an economic theory of management is needed. The purpose of this paper is to supply one. Managerial control will be treated as an intermediate product of the firm. At every administrative level, control from above is used as an input and combined with labor to produce supervision or control for the next lower level. This simple idea will be worked out with that old war horse, the Cobb–Douglas production function, to derive some results which are simple and yet interesting. A solution will be given to the problems of the optimal size of firms, the optimal structure of its administration, and in particular, the optimal span of control.

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2. The Production Function

Consider an organization with well-defined administrative levels $r \quad 1, \dots, R$, where R denotes the presidential or top level. Let $r = 0$ be the level of production workers or operatives.

How shall we describe the output of managers? Managers at level r produce "supervision" or "managerial control" y_r of managers at the next lower level $r - 1$. They produce this managerial control with their own labor x_r and with supervision by their own superiors y_{r+1}.

Each office is a black box whose inputs are, respectively, 1 unit of labor, y_{r+1}/x_r units of supervision by the next level of managers.

The output per office at hierarchical level r is thus

$$x_r f_{r+1}\left(\frac{y_{r+1}}{x_r}\right) = y_r . \tag{2.1}$$

Through successive substitution

$$y_0 = x_0 f_1\left(\frac{x_1}{x_0} f_2\left(\frac{x_2}{x_1} f_3\left(\cdots f_R\left(\frac{x_R}{x_{R-1}}\right)\cdots\right)\right)\right).$$

Here y_0 is management's output of "effective labor units" created through inputs of operative labor x_0 and of supervisory labor $x_1, \dots, x_R$ at the various supervisory levels r.

In a slightly more general formulation output y_r at each level is a function of labor and supervisory input for that level

$$y_r = F_{r+1}(x_r, y_{r+1}).$$

Operative labor in efficiency units produced by management is then

$$y_0 = F_1(x_0, F_2(x_0, \dots, F_R(x_{R-1}, x_R), \dots)). \tag{2.2}$$

Final output is the result of a combination of this effective labor y_0 with capital (machines) z.

$$q = z f_0\left(\frac{y_0}{z}\right).$$

Machines are thus in the same relationship to workers as subordinates are in relation to supervisors.

The production function is linear homogeneous

$$q = z f_0\left(\frac{x_0}{z} f_1\left(\frac{x_1}{x_1} f_2\left(\cdots x_{R-1} f\left(\frac{x_R}{x_{R-1}}\right)\cdots\right)\right)\right) \tag{2.3}$$

in the variables $z, x_0, x_1, \dots, x_R$.

The more general production function

$$q = F_0(z, F_1(x_0, F_2(\cdots F_R(x_{R-1}, x_R) \cdots))) \tag{2.4}$$

is linear homogeneous when all the F_r are linear homogeneous. If, however, the presidential input is fixed at unity

$$x_R = 1,$$

the production function for the organization in terms of the variable inputs $z, x_0, x_1, \ldots, x_{R-1}$ is not homogeneous.

From now on we specify the functions F_r to be Cobb–Douglas

$$\begin{aligned} F_0 &= b_0 z^{\gamma} y_0^{\delta}, \\ F_r &= b(r)\, x_{r-1}^{\alpha} y_r^{\beta}, \qquad r = 1, \ldots, R-1. \end{aligned} \tag{2.5}$$

Notice that we assume the output elasticities α and β to be the same at all administrative levels. This may be justified by arguing that administration is administration no matter at what level. We allow, however, the output quantities $b(r)$ per unit inputs to be different at different levels. Now the composite production function of the organization is

$$q = b_0 b(1)^{\delta} b(2)^{\delta\beta} \cdots b(R)^{\delta\beta^{R-1}} z^{\gamma} x_0^{\alpha\delta} x_1^{\alpha\delta\beta} \cdots x_{R-1}^{\alpha\delta\beta^{R-1}}$$

or

$$q = b_R \cdot z^{\gamma} \prod_{r=0}^{R-1} x_r^{\alpha\delta\beta^r}, \tag{2.6}$$

where

$$b_R = b_0 \prod_{r=1}^{R} b(r)^{\delta\beta^{r-1}}.$$

The composite Cobb–Douglas production function (2.6) is homogeneous of degree η_R in the variable inputs $z, x_0, \ldots, x_{R-1}$ where

$$\eta_R = \gamma + \alpha\delta \sum_{r=0}^{R-1} \beta^r = \gamma + \alpha\delta(1 - \beta^R)/(1 - \beta). \tag{2.7}$$

When all the F_r are linear homogeneous, $\alpha + \beta = 1$, $\gamma + \delta = 1$, then

$$\eta_R = 1 - \delta + \delta(1 - \beta^R) = 1 - \delta\beta^R. \tag{2.8}$$

The degree of homogeneity of the production function is less than one and depends on the organization's top level R (to be called its organizational level). The degree of homogeneity approaches unity as the organizational level is increased.

3. EFFICIENCY

In this section we consider questions of efficiency for organizations. By efficiency we mean that the organization's cost of delivering a given output q is minimized. Alternatively, for a given budget C, the organization's output is maximized.

Since we are not concerned with the external problems of market structure, we assume perfect competition in the markets for output, for labor of various administrative ranks r and for capital.

An important assumption is that salary of staff depends only on the administrative level r. This means we either ignore salary increments due to length of service, or we consider a "representative" holder of a position of rank r whose salary contains an average component due to length of service. But the principal part of salaries must be assumed to be determined by rank.

Efficiency will be studied in terms of the changes that efficient organizations should make to adjust their output to varying demand.

3.1. In the very short run, only labor input at the operative level $r = 0$ can be changed. The production function is then, in effect

$$q = \tilde{q} \cdot (x_0/\tilde{x}_0)^{\alpha\delta}, \tag{3.1}$$

where $\tilde{q}_0$ was the original output and $\tilde{x}_0$ the associated input of operative labor. This is an ordinary Cobb–Douglas function with an output elasticity of operative labor $\alpha\delta < 1$. Cost is then

$$C = \tilde{C} + w_0(x_0 - \tilde{x}_0)$$

or

$$C(q) = \tilde{C} + w_0\tilde{x}_0 \cdot [(q/\tilde{q})^{1/\alpha\delta} - 1], \tag{3.2}$$

$$C(q) = F_0 + w_0\tilde{x}_0(\tilde{q})^{-1/\alpha\delta} \cdot q^{1/\alpha\delta} \tag{3.3}$$

$$= F_0 + C_0 q^{1/\alpha\delta},$$

where F_0 denotes fixed cost, i.e., all cost other than operative labor cost, and C_0 a proportionality factor.

3.2. In the short-run proper, the organization can change staff inputs at all levels r except the president's, $r = 0,\ldots, R - 1$. The organization's production function becomes

$$q = \tilde{q} \prod_{r=0}^{R-1} (x_r/\tilde{x}_r)^{\alpha\delta\beta^r} \tag{3.4}$$

and its cost is

$$C = \tilde{C} \sum_{r=0}^{R-1} w_r(x_r - \tilde{x}_r) = F_1 + \sum_{r=0}^{R-1} w_r x_r. \tag{3.5}$$

In order to minimize the cost C of producing a given output q consider the Lagrangian

$$F_1 + \sum_{r=0}^{R-1} w_r x_r + \lambda \left[q - \tilde{q} \prod_{r=0}^{R-1} (x_r/\tilde{x}_r)^{\alpha\delta\beta^r} \right].$$

Differentiation yields:

$$w_r - \lambda \tilde{q} \prod_{i=0}^{R-1} (x_i/\tilde{x}_i)^{\alpha\delta\beta^r} \cdot (\alpha\delta\beta^r/x_r) = 0,$$

from which, using the production function (2.6)

$$x_r = \lambda \cdot q(\alpha\delta\beta^r/w_r). \tag{3.6}$$

(Since the Lagrangian is a convex function in the x_r, the condition (3.6) is both necessary and sufficient.)

Staff inputs at every level should thus be made proportional to output. We ignore the fact that the x_r must be integers and treat all labor, except the president's, as continuously variable. At the higher levels, when x_r is small, the resulting x_r of the analysis are only approximations.

Staff should also be proportional to output elasticities $\alpha\delta\beta^r$ and inversely proportional to salaries w_r. The latter fact implies that salary expenditure at the various levels

$$w_r x_r = \lambda q \alpha\delta\beta^r \tag{3.7}$$

should be proportional to output elasticities $\alpha\delta\beta^r$. Since $0 < \beta < 1$, this means that the salary bill for a level r decreases exponentially with that level.

The number of immediate subordinates per administrator is often called the span of control. The average span of control for level r is

$$s_r = x_{r-1}/x_r. \tag{3.8}$$

Now formula (3.6) implies that for cost to be minimized the average span of control should be

$$s_r = (1/\beta) \cdot (w_r/w_{r-1}). \tag{3.9}$$

This interesting formula may be read in two ways:

$$w_r = \beta s_r w_{r-1}. \tag{3.10}$$

Since $ws_r x_{r-1}$ is the combined salary of those supervised (on the average) by an administrator at level r, (3.10) states that the span of control should establish a proportional relationship between salaries of supervisors and those of the supervised [4].

The relationship (3.9) is particularly interesting in those cases where salaries form a geometric progression.

$$w_r = w_0 g^r, \qquad r = 0, \ldots, R-1, \tag{3.11}$$

where we except again the president. This is the rule in many organizations [5].

The ancient Artaśastra of Kautilia already contains a salary scale (3.11) with $g = 2$.

In combination with (3.11), Eq. (3.9) implies a constant average span of control s at all levels

$$s_r = s = g/\beta. \tag{3.12}$$

This optimal span of control is proportional to the incremental salary factor g and inversely proportional to the output elasticity of supervision β.

Substitute now the recommended inputs (3.6) into the production and cost functions (3.4) and (3.5) in order to obtain the short-run cost curve, that is, cost as a function of output when all labor is adjusted optimally.

$$C(q) = F_1 + C_1 q^{(1-\beta)/\alpha\delta(1-\beta^R)}, \tag{3.13}$$

where F_1 is fixed cost and C_1 depends on the initial output $\tilde{q}$ and the initial labor inputs $\tilde{x}_r$.

A comparison of (3.3) and (3.13) shows that the elasticity of variable cost is smaller in the short run than in the very short run

$$\frac{1-\beta}{\alpha\delta(1-\beta^R)} < \frac{1}{\alpha\delta}. \tag{3.14}$$

This means that a given change in output requires a smaller change in cost in the short run compared to the very short run. The difference increases with R. In the case of linear homogeneous Cobb–Douglas functions (2.7) one has the simpler elasticities

$$\frac{1}{\delta(1-\beta^R)} > \frac{1}{(1-\beta)\,\delta} = \frac{1}{\alpha\delta}. \tag{3.15}$$

Notice that our analysis in no way assumes that the initial input $\tilde{x}_r$ were optimally chosen or that salaries were of the form (3.11). In fact, the levels of the initial inputs affect both the fixed costs F_0, F_1 and the proportionality factors C_0 and C_1.

3.3. In the medium run, the level of capital stock z may be adjusted as well. An analysis along the lines of Section 3.2 shows that the average span of control must again satisfy (3.9) and that in addition the capital-labor ratio should be set at

$$\frac{z}{x_0} = \frac{\gamma}{\alpha\delta} \cdot \frac{w_0}{k}, \tag{3.16}$$

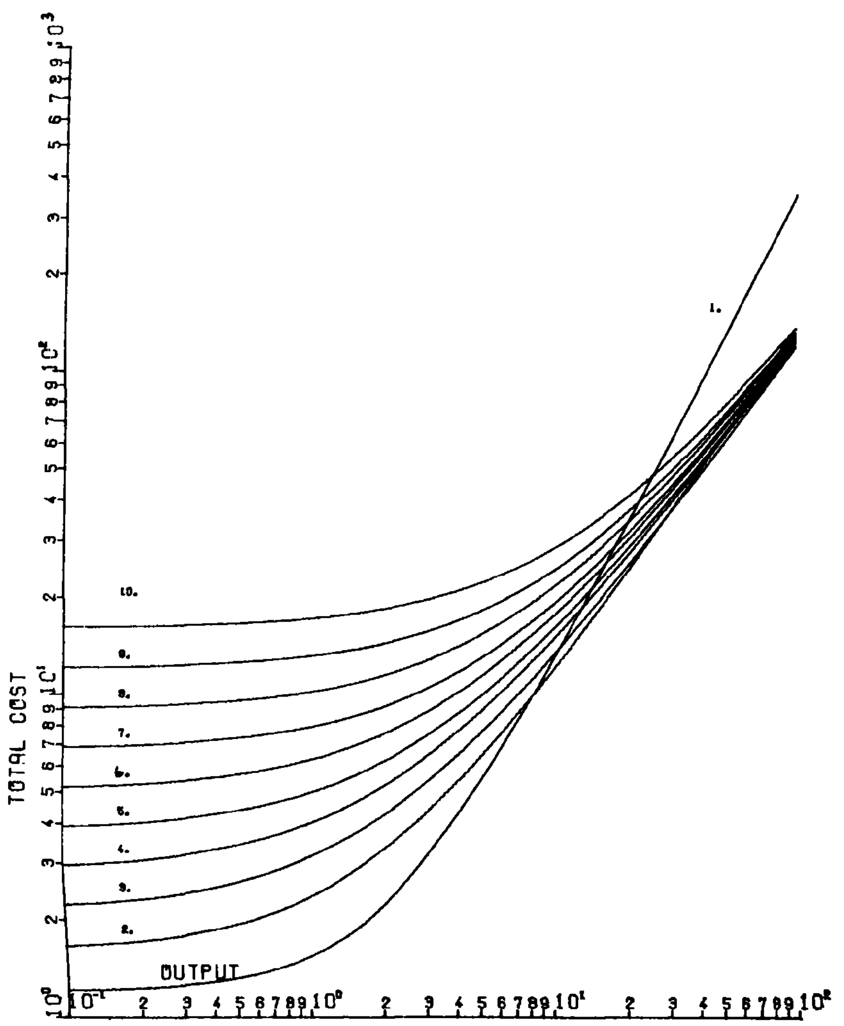

FIG. 1. Cost curves for different R (medium run).

where k is the price (interest plus depreciation) of capital services. The medium-run cost function has the form

$$C(q) = F_2 + C_2 \cdot q^{(1/(\gamma + \alpha\delta(1-\beta^R)/(1-\beta)))}. \tag{3.17}$$

In the linear homogeneous case this simplifies to

$$C(q) = F_2 + C_2 \cdot q^{1/(1-\delta\beta^R)}, \tag{3.18}$$

where the fixed cost consists now of only the president's salary w_R (Fig. 1). The elasticity is again smaller than that for the short-run cost function

$$\frac{1}{1-\delta\beta^R} < \frac{1}{\delta(1-\beta^R)} \quad \text{since} \quad 0 < \delta < 1.$$

The proportionality factor C_2 in (3.18) depends again on the initial choices $\tilde{z}, \tilde{x}_0, \ldots, \tilde{x}_{R-1}$ of inputs.

In the long run the firm can vary everything including the level R of its organizational structure. From the general form of the cost functions (3.16) it may be inferred that, no matter how large the fixed cost F_2 or whatever the proportionality factor C_2 may be, if output is sufficiently large, it pays to increase the organizational level R so as to reduce the elasticity $1/(1-\delta\beta^R)$ of the cost curve.

This general conclusion can be made more precise by a consideration of the minimum points on the average cost curves, which is the subject of the next section.

4. MINIMUM AVERAGE COST

4.1. The average cost of producing output q is of the form

$$\bar{C}(q) = (F/q) + c \cdot q^{\eta-1}. \tag{4.1}$$

Now in the linear homogeneous case (2.7)

$$\begin{aligned} \eta &= 1/\delta(1-\beta) && \text{in the very short run,} \\ &= 1/\delta(1-\beta^R) && \text{in the short run,} \\ &= 1/(1-\delta\beta^R) && \text{in the medium run.} \end{aligned}$$

Since $\eta > 1$ in any case, the average cost curve $\bar{C}(q)$ must be U-shaped.

Now in the medium run, when all inputs are variable but R is given, the proportions of all inputs are fixed through cost minimization. The scale of output is then determined by choosing the level of any one input, for instance, the number of subordinates x_{R-1} of the president, the president's span of control

$$s_R = x_{R-1}. \tag{4.2}$$

In terms of this variable, output q and cost C_2 turn out to be as follows: From (3.9) we have

$$\frac{x_r}{s_R} = \frac{w_{R-1}}{w_r} \cdot \beta^{r-R+1}, \tag{4.3}$$

and from (3.16),

$$\frac{z}{s_R} = \frac{\gamma}{\alpha\delta} \cdot \frac{w_{R-1}}{k} \beta^{-R+1}. \tag{4.4}$$

Substituting in the production function (2.6)

$$\begin{aligned} q &= b_R \cdot \left(\frac{\gamma}{\alpha\delta} \frac{w_{R-1}}{k} \beta^{-R+1} s_R\right)^{\gamma} \prod_{r=0}^{R-1} \left(\frac{w_{R-1}}{w_r} \beta^{r-R+1} s_R\right)^{\alpha\delta\beta^r}, \\ q &= d_R \cdot s_R^{1-\delta\beta^R}, \end{aligned} \tag{4.5}$$

where

$$d_R = b_R \beta^{1-R} \left(\frac{\gamma}{\alpha\delta k}\right)^{\gamma} w_{R-1}^{1-\delta\beta^R} \prod_{r=0}^{R-1} w_r^{-\alpha\delta\beta^r}. \tag{4.6}$$

The total cost is

$$\begin{aligned} C &= w_R + kz + \sum_{r=0}^{R-1} w_r x_r, \\ C &= w_R + s_R w_{R-1} \cdot \frac{1-\delta\beta^R}{\alpha\delta} \cdot \beta^{1-R} \end{aligned} \tag{4.7}$$

using (4.3), (4.4).

Average or unit cost of output in terms of s_R is, therefore,

$$\bar{C}_R = \frac{w_R}{d_R} \cdot s_R^{\delta\beta^R - 1} + \frac{w_{R-1}}{d_R} \frac{1-\delta\beta^R}{\alpha\delta} \beta^{1-R} \cdot s_R^{\delta\beta^R}. \tag{4.8}$$

Here the first term decreases from infinity to zero and the second term increases from zero to infinity, as s_R is increased from zero. Average cost must, therefore, have a unique finite minimum at a positive level s_R.

To obtain this minimum, differentiate (4.8) and equate to zero

$$(\delta\beta^R - 1)\, w_R s_R^{\delta\beta^R - 2} + \delta\beta^R w_{R-1} \frac{1-\delta\beta^R}{\alpha\delta} \beta^{1-R} \cdot s_R^{\delta\beta^R - 1} = 0$$

(where we have cancelled d_R) or

$$s_R = (\alpha/\beta) \cdot (w_R/w_{R-1}). \tag{4.9}$$

This formula for the optimal presidential span of control should be compared with that for other ranks

$$s_r = (1/\beta)(w_r/w_{r-1}). \tag{3.9}$$

In order to minimize the average cost of output, the president's span of control should be made smaller by a factor α compared to any subordinate's span of control for the same salary ratio. The burden of the president is thus made less, *ceteris paribus*, in order to compensate for the fact that his task is not aided by supervision.

Of course, support may come from staff that is directly responsible to an office and is not part of the line-organization. A natural extension of the production function (2.1) is

$$y_r/x_r = f(y_{r+1}/x_r, z_r/x_r), \tag{4.10}$$

where z_r is the staff for offices at level r, but this will not be pursued here.

Let the salary structure be of the form

$$\begin{aligned} w_r &= w_0 g^r, \qquad r = 0, \dots, R-1, \\ w_R &= w_0 \cdot g^{R-1} h \end{aligned} \tag{4.11}$$

allowing for a different salary step at the president's level. The spans of control are then

$$\begin{aligned} s_r &= g/\beta, \qquad r = 1, \dots, R-1, \\ s_R &= (\alpha/\beta) \cdot h. \end{aligned} \tag{4.12}$$

The president's span of control under minimization of average cost is now larger (or smaller) when αh is larger (or smaller) than g.

5. The Long Run and Returns to Scale

Consider now a long-run change in the size of the organization. Specifically let the level R be increased to $R+1$ and let inputs be changed in such a way that average cost is again minimized.

The analysis of 4 shows that this means that a layer has been added to the organization's structure at the bottom using the same span of control

$s = 1/\beta$. This does not imply, of course, that all operative labor is newly hired and that everybody else has been promoted by one rank. Typically it means instead that all inputs, including capital, have been expanded by the same factor s and that one additional layer of $s_{R+1} = (\alpha/\beta) \cdot h$ administrators has been added below the new presidential level $R + 1$.

How does this change affect minimum average cost? For an organization of R levels total cost is, using (4.7),

$$C_R = w_R + \frac{\alpha}{\beta} \frac{w_R}{w_{R-1}} \cdot w_{R-1} \cdot \frac{1 - \delta\beta^r}{\alpha\delta} \cdot \beta^{1-R} = w_R \cdot \frac{1}{\delta\beta^R} \cdot$$

The input quantities are

$$x_r = (w_R/w_r)\, \alpha\beta^{r-R},$$
$$z = (\gamma/\alpha\delta)(w_R/k)(\alpha/\beta)\, \beta^{-R+1} = (w_R/k)\, \delta^{-1}\beta^{-R}.$$

Substituting

$$q_R = b_R(\beta^{-R}w_R)^{1-\delta\beta^R}\, (k\delta)^{-\gamma} \prod_{r=0}^{R-1} (\alpha\beta^r/w_r)^{\alpha\delta\beta^r}.$$

A straightforward calculation yields then the average cost

$$\bar{C}_R = \frac{C_R}{q_R} = \frac{1}{b_R}\, \alpha^{\delta(1-\beta^R)}\delta^{-\delta}\beta^{-\delta(\beta/\alpha)(1-\beta^R)} \cdot k^{\delta} w_R^{\delta\beta^R} \cdot \prod_{r=0}^{R-1} w_r^{\alpha\delta\beta^r}. \tag{5.1}$$

With the salary scale (4.11) the spans of control are (4.12) and the total wage bill is

$$\begin{aligned} W_R &= w_R + s_R \sum_r s^{R-1-r} \cdot w_0 g^r \\ &= w_0 h g^{R-1} + (\alpha/\beta)\, h \sum_{r=0}^{R-1} (g/\beta)^{R-1-r} g^r w_0 \qquad (5.2) \\ &= w_0 h g^{R-1} \cdot (1/\beta^R) = (w_R/\beta^R). \end{aligned}$$

The president's salary represents then a fraction β^R of the total wage bill. This fraction decreases with the number of administrative levels R.

Total production cost including capital cost may be calculated as

$$C_R = (w_R/\delta\beta^R) = (1/\delta)\, w_R\,. \tag{5.3}$$

The wage bill is then the constant part δ of total cost, a consequence of our Cobb-Douglas assumptions about the production function. The output that minimizes average cost is

$$
\begin{aligned}
q_R &= b_R\left((\gamma/\alpha\delta)(w_0/k)\, s_R s^{R-1}\right) \prod_{r=0}^{R-1} (s_R s^{R-1-r})^{\alpha\delta\beta^r}, \\
q_R &= b_R\left((\gamma/\alpha\delta)(w_0/k)\right)^{\gamma} s_R^{1-\delta\beta^R} s^{R-1+(\delta/\alpha)\beta^R-(\delta/\alpha)\beta}.
\end{aligned}
\tag{5.4}
$$

5.2. For more concrete results assume now

$$b(r) = b, \qquad r = 1,\dots, R$$

so that from (2.6)

$$b_R = b_0 \cdot b^{\delta/(1-\beta)(1-\beta^R)}.$$

Average cost is then

$$
\begin{aligned}
\bar{C}_R &= \frac{w_0{}^{\delta} k^{\gamma}}{b_0 b^{\delta/\alpha} g^{\delta\beta/\alpha} \alpha^{\delta} \delta^{\delta} \gamma^{\gamma} \beta^{\delta\beta/\alpha}} \left[b^{1/\alpha} \left(\frac{\beta}{g}\right)^{\beta/\alpha} \frac{h\alpha}{g}\right]^{\delta\beta^R}, \\
\bar{C}_R &= c \cdot m^{\delta\beta^R} \qquad \text{(say).}
\end{aligned}
\tag{5.5}
$$

With increasing R, $\delta\beta^R$ falls to zero.

The minimum average cost of production approaches a constant value c, as R grows large. (This is also true when the $b(r)$ are not constant but are only approximately constant $\lim_{r\to\infty} b(r) = b$.) The limiting value of minimum average cost is

$$c = w_0{}^{\delta} k^{\gamma} b_0^{-1} b^{-\delta/\alpha} g^{-\delta\beta/\alpha} \alpha^{-\delta} \beta^{-\delta\beta/\alpha} \gamma^{-\gamma} \delta^{-\delta}. \tag{5.6}$$

It depends on all the parameters of the production functions, on capital cost and on the wage structure, but not on presidential salaries.

Thus in the limit, organizational size does not matter and large firms are as efficient as very large firms.

But what about the cost level of small firms compared to large ones? This depends critically on the value of m. Small firms have lower average cost if and only if

$$m = \frac{h}{g}\alpha \cdot b^{1/\alpha} g^{-\beta/\alpha} < 1. \tag{5.7}$$

To understand the meaning of $m \gtrless 1$, compare the productivity measures $y_r/x_r = e_r$ at the various levels r. From the production function (2.1) for constant returns (2.7)

$$\begin{aligned} e_r &= y_r/x_r = by_{r+1}^{\beta}x_r^{-\beta} \\ &= b(y_{r+1}/x_{r+1})^{\beta} \cdot (x_{r+1}/x_r)^{\beta} \\ &= be_{r+1}^{\beta} \cdot s^{-\beta}. \end{aligned}$$

Now $be_{r+1}^{\beta}s^{-\beta} = e_r \gtrless e_{r+1}$ whenever $e_{r+1} \lessgtr b^{1/\alpha}s^{-\beta/\alpha}$ so that productivity increases from the president on down whenever

$$e_{R-1} < b^{1/\alpha}s^{-\beta/\alpha}. \tag{5.8}$$

But

$$e_{R-1} = bs_R^{\beta}$$

and $s_R = s \cdot \alpha h/g$ from (4.12).

Substituting in (5.8) yields $m > 1$. We have shown that as efficiency increases with R, minimum average cost decreases with R. The superiority of large organizations over small ones depends on the organization's span of control. Small organizations have higher average costs whenever

$$s_R > (bs)^{1/\alpha}, \tag{5.9}$$

i.e., when the president's span of control is large enough to push productivity at the next level below (5.8).

Now the spans of control s and s_R depend in turn on the salary structure. Using (4.11) and (4.12) the condition (5.7) for increasing returns to scale becomes

$$h > [(g/b)\,\alpha^{-\alpha}\beta^{-\beta}]^{1/\alpha}.$$

Whether there are increasing or decreasing returns to scale in administration is thus critically dependent on presidential salaries. Suppose for a moment that there are increasing returns to scale; then there will be a tendency for fewer but larger firms to evolve. Competition for presidential positions will tend to drive h down until $m = 1$ or

$$h = [(g/b)\,\alpha^{-\alpha}\beta^{-\beta}]^{1/\alpha}. \tag{5.10}$$

This argument shows that the presidential salaries will tend to be in such a proportion to salaries of other ranks that returns to scale in administration are approximately constant. Then there are no economies of scale in administration that distinguish larger firms from smaller ones.

The following proposition is not difficult to show. Let management technology be described by a Cobb–Douglas production function

$$y_{r+1} = b x_r^{\alpha} y_{r+1}^{\beta}, \qquad \alpha + \beta = 1 \tag{5.11}$$

with the same coefficients b, β at all levels. Then the only wage system that generates a minimum average cost c which is independent of R is of the form

$$w_r = w_0 g^r, \qquad r = 0, 1, \ldots, R - 1,$$

$$w_R = w_0 h g^{R-1}, \qquad h = [(g/b) \cdot \alpha^{-\alpha} \beta^{-\beta}]^{1/\alpha}.$$

With constant returns to scale (for outputs q_R that minimize average cost) one has, of course,

$$q_R = q_0 s_R s^{R-1} \tag{5.12}$$

so that output grows exponentially with the number of levels. It is remarkable that the question of the returns to scale in administration should depend only on presidential salaries relative to those of other ranks as stated in (5.10).

6. Profit Maximization and the Optimal Size of Firms

When the organization considered is a business firm, it is appropriate to consider profit maximization. It is useful to distinguish between owner-operated firms, whose owner is the president and receives all profits, and corporations with salaried presidents receiving no share of profits. Of course, intermediate situations will occur.

In the medium-run profits are a function of the input variables z, $x_0, \ldots, x_{R-1}$, which may be freely chosen.

$$G = pq(z, x_0, x_1, \ldots, x_{R-1}) - kz - \sum_{r=0}^{R-1} w_r x_r - w_R. \tag{6.1}$$

With the Cobb–Douglas function (2.11) we have

$$G = p b_R z^{\gamma} \prod_{r=0}^{R-1} x_r^{\alpha\delta\beta^r} - kz - \sum_{r=0}^{R-1} w_r x_r - w_R. \tag{6.2}$$

This is a concave function in the variables $z, x_0, \ldots, x_{R-1}$. For a maximum the necessary and sufficient conditions are

$$\partial G/\partial z = 0, \qquad \partial G/\partial x_r = 0, \qquad r = 0, \ldots, R - 1. \tag{6.3}$$

(Zero levels of the variables need not be considered.) Condition (6.3) yields at once

$$z = \gamma pq/k, \qquad x_r = \alpha\delta\beta^r pq/w_r. \tag{6.4}$$

These conditions are identical with (3.6) when one sets

$$\lambda = p. \tag{6.5}$$

Thus profit maximization implies cost minimization, as is well known. The λ of Section 3 may now be interpreted as marginal cost, and Eq. (6.4) states that marginal cost equals price.

The maximum profits thus obtained are

$$\begin{aligned}\pi_R &= pq - \gamma pq - \sum_{r=0}^{R-1} \alpha\delta\beta^r pq - w_R \\ &= pq(1 - \eta_R) - w_R \\ \pi &= \delta\beta^R pq - w_R.\end{aligned}$$

Call R profits before presidential remuneration gross profits. Gross profits as a percentage of sales are $\delta\beta^R$. This profit margin decreases with the organizational level R.

For more detailed results it is necessary to determine the optimum output q or supply function of the firm. Substitute (6.4) in the production function (2.6)

$$q = b_R(pq\gamma/k)^\gamma \prod_{r=0}^{R-1} (pq\alpha\delta\beta^r/w_r)^{\alpha\delta\beta^r}. \tag{6.6}$$

From this

$$\begin{aligned}q &= \gamma^{(\gamma/\delta)\beta^{-R}} \cdot (\alpha\delta)^{\beta^{-R}-1}\, \beta^{(\beta/\alpha)-R} b(R)^{(1/\delta)\beta^{-R}}\, p^{(1/\delta)\beta^{-R}-1} k^{-(\gamma/\delta)\beta^{-R}} \prod_{r=0}^{R-1} w_r^{-\alpha\beta^{r-R}} \\ &\sim p^{(1/\delta)\beta^{-R}-1} k^{-(\gamma/\delta)\beta^{-R}} \prod_{r=0}^{R-1} w_r^{-\alpha\beta^{r-R}}.\end{aligned} \tag{6.7}$$

The elasticities of supply with respect to output price and to the various factor prices are constant. The price elasticities increase with R to infinity. Of course, the supply function is homogeneous of degree zero with respect to the level of all prices. Assume again the wage structure (4.11) and production function (2.6) with $b_r = b$, $r = 1,\ldots, R-1$. Then the supply function assumes the form

$$q_R = b_0 b^{(\delta/\alpha)(1-\beta^R)}(\gamma/k)^{\gamma/\delta\beta^R} \cdot p^{(1-\delta\beta^R)/\delta\beta^R} \cdot (\alpha\delta/w_0)^{\beta^{-R}-1} \cdot (\beta/g)^{-R+(\beta^{1-R}/\alpha)-(\beta/\alpha)}. \tag{6.8}$$

The elasticities with respect to price p, basic wage w_0, and capital cost k are, respectively,

$$(1/\delta\beta^R) - 1, \qquad 1 - (1/\beta^R), \qquad \text{and} \qquad -(\gamma/\delta\beta^R).$$

All elasticities increase (in absolute value) as R is increased.

Substituting (6.8) in the profit function (6.6) we obtain the value of gross profits

$$\begin{aligned} G_R &\quad \delta\beta^R p q_R, \\ G_R &\quad w_0 h^{-1} \gamma s^{\beta} \gamma^{-1} \cdot g^R \cdot (p/c)^{\beta^{-R}}, \end{aligned} \tag{6.9}$$

where c is the limiting value of minimum average cost for large firm (5.6). The ratio p/c is critical for the dynamics of the industry.

When prices are high enough so that $p > c$ then profits increase with size faster than g^R. No matter how high one sets the constant in the presidential salary formula (4.12) the factor $a^{\beta^{-R}}$ exceeds this for some R and goes to infinity as β^{-R} increases indefinitely with R. (Recall that $\beta < 1$.) Hence, the optimal size is the maximal size. There is a premium on rapid growth. This expansion of firms in the industry combined with a stationary demand will cause prices to fall until $p = c$.

Consider next that $p < c$. Then the profit function (6.9) has a finite maximum with respect to R. This is true because g^R in (6.9) is increasing and $(p/c)^{\beta^{-R}}$ is decreasing with R and the last effect eventually dominates. Treating R as a continuous variable the optimal level R is determined as

$$R = \frac{\log(\log g\,(\log(p/c) \cdot \log\beta))}{|\log\beta|}. \tag{6.10}$$

Since R is an integer, this formula (6.10) applies only approximately.

Now gross profits are

$$\begin{aligned} G_R &= w_0 h g^{R-1} \cdot (1/m) \cdot (p/c)^{\beta^{-R}} \\ &= w_R \cdot (1/m)(p/c)^{\beta^{-\hat{R}}}, \end{aligned}$$

where $\hat{R}$ is the optimal level of R.

If

$$m \leq (p/c)^{\beta^{-\hat{R}}} \tag{6.11}$$

then profits will be adequate to pay a presidential salary. Otherwise, firms will have to be sold to private owner–managers or, failing that, will have to shut down.

In the latter event, supply will decrease and prices will be raised to the point where the condition (6.11) for adequate executive compensation is met or $p = c$, prices equal the limiting value of minimum average cost (to be considered next).

We conclude that a long-run equilibrium is possible with a well-defined optimum size of the firm (6.10) provided presidential salaries are low enough to meet the conditions (6.11) or provided owner–managers are satisfied with a compensation below that of corporate presidents for the same size of firm.

The final case to be considered is $p = c$, prices equal the limiting value of minimum average cost.

Then from (6.9)

$$\begin{aligned} G_R &= w_0 b^{-1/\alpha} s^{\beta/\alpha} \cdot \alpha^{-1} g^R \\ &= w_0 (h/g) \cdot (1/m)\, g^R \\ &= (1/m)\, W_R . \end{aligned} \tag{6.13}$$

Now in equilibrium it must be that $m = 1$: No profits remain after payment of presidential salaries.

For suppose that $m > 1$, so that net profits are positive. Then top management will try to augment its compensation. Firms have an incentive to grow and new firms to enter the industry raising the demand for presidents. The result is either an increase of h and hence m or a fall of prices and a return to Case 2.

Suppose, on the other hand, that $m < 1$. Then firms' profits are insufficient to pay presidential salaries. They must either leave the industry or be sold to private owners who are content with smaller compensation than corporate presidents.

As firms are converted to single ownership, the demand for corporation presidents falls and h is adjusted downward. If h reaches the point where $m = 1$, then formula (6.13) shows that corporate firms with salaried presidents have net profits of zero.

Otherwise, it happens again that all firms are owned by owner–managers accepting compensation below salary levels of corporate presidents. But now returns to scale are constant so that privately owned firms of any organizational size may coexist.

Long-run equilibrium of an industry whose firms include those run by salaried presidents implies that both $p = c$ and $m = 1$, net profits are zero and presidents are paid equilibrium wages, average costs are at the minimum level c, and returns to scale are constant so that firms can achieve these two goals at all levels R of organizational size.

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Weak Path Independence*

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The principal problem of social choice is that it appears to be impossible to find a method of aggregating individual preferences into a social choice function (or social preference relation) simultaneously satisfying a small number of "reasonable" criteria. Some of these criteria, namely the Pareto principle and the principle that "power" ought to be distributed somewhat equally (e.g., there should not be a dictator) are clearly normative. On the other hand, the independence of irrelevant alternatives seems to be a "minimal" technical requirement for any choice process (see Plott [1]). The status of the various collective rationality conditions (e.g., transitivity) is less clear: is some such requirement necessary for the choice to be "well defined" (see Sen [2, p. 50ff]) or is its justification primarily normative?

Throughout this paper we assume that a society is composed of n people, each of whom has a preference relation R_i which is a weak order on S, where S is the finite set of possible social states. The social choice from a set V where V is a subset of S is denoted $C(V, R_1, R_2, \ldots, R_n)$ or $C(V)$ where the dependence on the preferences of the individuals is unambiguous. $C(V)$ is assumed to be nonempty and contained in V. We utilize the following standard definitions.

P (Pareto Principle): $\forall i \; xP_iy \Rightarrow \{x\} = C(\{x, y\})$.

I (Independence of Irrelevant Alternatives): Let $(R_1, \ldots, R_n)$ and $(R_1', \ldots, R_n')$ be two sets of individual preferences. $\forall V \subseteq S$ if

$\forall x, y \in V, \forall i \; xR_iy \Leftrightarrow xR_i'y$, then $C(V, R_1, \ldots, R_n) = C(V, R_1', \ldots, R_n')$.

D (Nondictatorship): $\nexists i$: $\forall x, y \in S \quad xP_iy \Rightarrow \{x\} = C(\{x, y\})$.

It seems obvious that the normative requirements of nondictatorship and Pareto are already extremely weak versions of widely held ethical beliefs about "good" social choice procedures. And since we find con-

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vincing Plott's defense of independence as a minimal condition for social choice, we shall assume independence as a background condition throughout the present paper.

Much of the recent literature in social choice theory has focused on what Arrow [3] called the requirement of "collective rationality." Most of this work has been carried out in a somewhat different formal context from ours. For example, Arrow [3], Brown [4], and Mas-Collel and Sonnenschein [5] were concerned with finding a mapping from collections of individual preference relations to a binary relation which might be called the social preference relation. The connection between those theories and ours is, of course, through the theory of revealed preference. If the social choice function is uniquely rationalized (see Richter [6]) by a binary relation, we call this binary relation the social preference relation. Mas-Collel and Sonnenschein [5] have shown that if Arrow's collective rationality condition (that the social preference relation be a weak order) is weakened to require only that the social preference relation be quasi-transitive or acyclic, impossibility theorems may still be obtained by strengthening Arrow's other axioms somewhat.

Essentially, the results of Arrow, Mas-Collel and Sonnenschein, and Brown indicate that when a collective rationality requirement is combined with P and I, either the "distribution of power" in society is required to be quite asymmetric or social choice must be indecisive. Requiring that social preference be a weak order implies the existence of a dictator, quasi-transitivity yields an oligarchy, and acyclic social preference implies the existence of what Brown has called a "collegium" of privileged members of the polity. One is tempted to conclude that any sort of rationality requirement for social preference will conflict with the Pareto axiom (P), the requirement of independence of irrelevant alternatives (I), and slight strengthenings of the nondictatorship axiom (D). It may be asked, therefore: why must society choose rationally? Throughout this essay we maintain (as does Arrow in our interpretation) that a rationality requirement must stand or fall on normative grounds and proceed therefore to an examination of normative justifications of such conditions.

Plott [7] introduced axiomatizations of quasi-transitive and acyclic choice. In particular he demonstrated that a choice function has a unique, total, reflexive, quasi-transitive (acyclic) rationalization if and only if it satisfies path independence (IP) (or path independence*, IP*) and the extension axiom (E).

$$\text{IP:} \quad (\forall V) \quad V \subseteq S \qquad C(V) = C\Big(\bigcup_i C(V_i)\Big)$$

for any collection of subsets of S such that

$$\bigcup_i V_i = V.$$

$$\text{IP*:}\quad (\forall V)\quad V \subseteq S \qquad C(V) \subseteq C\left(\bigcup_i C(V_i)\right)$$

for any collection of $V_i \subseteq S$ such that

$$\bigcup_i V_i = V.$$

$$\text{E:}\quad (\forall V)\quad V \subseteq S \qquad [x \in C(\{x, y\})\quad \forall y \in V \text{ and } x \in V] \Rightarrow x \in C(V).$$

In discussing the reasons for a collective rationality requirement, Arrow presented the following argument: transitivity (of the social preference relation) will ensure the "independence of the final choice from the path to it" [3, p. 120]. It was this line of argument that Plott employed as the justification of IP. But Arrow went on to elaborate his objection to an intransitive choice as follows: "an intransitive social choice mechanism may... produce decisions that are clearly unsatisfactory" [3, p. 120]. It seems to us that if the latter remark is taken as a clarification of the former one, then Arrow's statement is not so much a justification of IP or IP*, but rather is a defense of the following axiom.

$$\text{*IP (weak path independence):}\quad (\forall V)\quad V \subseteq S \qquad C(V) \supseteq C\left(\bigcup_i C(V_i)\right)$$

for any collection of $V_i \subseteq S$ such that

$$\bigcup_i V_i = V.$$

One way of putting it is this: a satisfactory choice procedure is one that ensures that if the problem of choosing over V is broken up into choosing over subsets and then choosing over the remaining elements, the final choice from this procedure should still be in the choice set from V. That is, we do not want to end up in an unsatisfactory position because of the way the agenda was manipulated or because of the way the institutions we are dealing with work. As we read Arrow, the principal reason that society should choose transitively is to ensure that *IP will be satisfied. Indeed, it seems that Sen's reading of the passage in question is quite close to our own. He remarks that if the social preferences are transitive "it is guaranteed that one of the best alternatives will be chosen independently of path, given transitivity" [2, p. 48].

As Plott showed, IP and E lead to quasi-transitive choice. Parks [8] proved that IP* is equivalent to Sen's α (discussed in Section II below).

Blair [9] proved that if a choice function satisfies α and E, (which he called Generalized Condorcet), then it has an acyclic rationalization. Further it is easily shown that if a choice function satisfies IP (or IP*) then choice on pairs is quasi-transitive (or acyclic). Since E simply ensures that the binary relation deduced from choice on pairs rationalizes the choice function, one can see that the Mas-Collel and Sonnenschein results can be generalized by substituting IP (or IP*) for quasi-transitivity (or acyclicity).

Thus if we insist that a choice procedure satisfies IP*, then many institutions that are ethically appealing on intuitive grounds must be rejected. Among these institutions are simple majority rule, Borda systems, and many others. Apparently IP* (no less than rationality requirements) when combined with the other axioms on social choice amounts to requiring that an extremely asymmetric distribution of power exist in the society and this conclusion is what most people find unsettling. Since we regard *IP as a more attractive attribute of a choice procedure than IP*, we are led to investigate whether imposing it as a substitute for a collective rationality axiom still forces the society to have an asymmetric distribution of power.

The plan of the paper is as follows. In Section II we explore the relationships between *IP and some other axioms that have appeared in the literature, and we prove a possibility theorem with *IP used for the rationality requirement. In Section III we give some general properties of choice processes which satisfy *IP and show that a natural extension of the Pareto principle to sets with more than two elements results in an impossibility theorem. The conclusions of the paper are given in Section IV.

II. A Possibility Theorem For Weakly Path Independent Social Choice

Plott [1973] showed that IP is equivalent to the following axioms:

$$\text{IP}': (\forall V_1)\ V_1 \subseteq S\ (\forall V_2)\ V_2 \subseteq S\ C(C(V_1) \cup C(V_2)) = C(V_1 \cup V_2)$$

and

$$\text{IP}'': (\forall V_1)\ V_1 \subseteq S\ (\forall V_2)\ V_2 \subseteq S\ C(V_1 \cup C(V_2)) = C(V_1 \cup V_2).$$

It is natural to ask if similar equivalences hold for *IP type axioms. It is not hard to show that if S contains fewer than four elements, then *IP is equivalent to:

$${}^*\text{IP}': (\forall V_1)\ V_1 \subseteq S\ (\forall V_2)\ V_2 \subseteq S\ C(C(V_1) \cup C(V_2)) \subseteq C(V_1 \cup V_2).$$

The following example shows that in general the two axioms are different.

Let $S = \{x_1, x_2, x_3, x_4\}$ and let $C(\ )$ be as shown in the table below.

V	$C(V)$	V	$C(V)$
$\{x_1, x_2\}$	$\{x_1\}$	$\{x_1, x_2, x_3\}$	$\{x_1\}$
$\{x_1, x_3\}$	$\{x_1\}$	$\{x_1, x_2, x_4\}$	$\{x_1, x_4\}$
$\{x_1, x_4\}$	$\{x_1\}$	$\{x_1, x_3, x_4\}$	$\{x_1\}$
$\{x_2, x_3\}$	$\{x_2, x_3\}$	$\{x_2, x_3, x_4\}$	$\{x_2, x_3\}$
$\{x_2, x_4\}$	$\{x_2\}$		
$\{x_3, x_4\}$	$\{x_3\}$	$\{x_1, x_2, x_3, x_4\}$	$\{x_1\}$

$$C(\{x_i\}) = \{x_i\} \qquad i = 1, 2, 3, 4.$$

It is straightforward to check that the choice function satisfies *IP′, but

$$\begin{aligned} &C(C(\{x_4\}) \cup C(\{x_1, x_2, x_3\}) \cup C(\{x_2, x_4\})) \\ &\quad = C(\{x_1, x_2, x_4\}) = \{x_1, x_4\} \not\subseteq C(\{x_1, x_2, x_3, x_4\}) \end{aligned}$$

so that *IP does not hold. Thus if we wish to ensure that altering the agenda cannot allow an "unsatisfactory" element to be chosen, the stronger axiom is required. Note that in this example,

$$C(C(\{x_1, x_2, x_3\}) \cup \{x_2, x_4\}) \not\subseteq C(\{x_1, x_2, x_3, x_4\})$$

so that while the choice function does satisfy *IP′ it does not satisfy *IP″.

$$\text{*IP}'': (\forall V_1)\ V_1 \subseteq S\ (\forall V_2)\ V_2 \subseteq S\ C(V_1 \cup C(V_2)) \subseteq C(V_1 \cup V_2).$$

In fact we have the following theorem.

THEOREM 1. **IP and *IP″ are equivalent.*

Proof. Since V_1 may be expressed as the union of its separate elements and since we require the choice for a set to be nonempty, it is immediate that *IP ⇒ *IP″. Conversely, *IP″ implies that

$$\begin{aligned} C\left(\bigcup_{i=1}^{k} C(V_i)\right) &= C\left(\bigcup_{i=1}^{k-1} C(V_i) \cup C(V_k)\right) \\ &\subseteq C\left(\bigcup_{i=1}^{k-1} C(V_i) \cup V_k\right) = C\left(\left(\bigcup_{i=1}^{k-2} C(V_i) \cup V_k\right) \cup C(V_{k-1})\right) \\ &\subseteq C\left(\bigcup_{i=1}^{k-2} C(V_i) \cup V_k \cup V_{k-1}\right) \\ &\ \ \vdots \\ &\subseteq C\left(\bigcup_{i=1}^{k} V_i\right). \end{aligned}$$

Q.E.D.

Sen [2, 10] discussed two rationality requirements for social choice which he termed properties α and β.

$$\alpha\colon x \in V_1 \subset V_2 \Rightarrow [x \in C(V_2) \Rightarrow x \in C(V_1)],$$
$$\beta\colon [x, y \in C(V_1) \text{ and } V_1 \subset V_2] \Rightarrow [x \in C(V_2) \Leftrightarrow y \in C(V_2)].$$

The interpretation of these properties given is that "property α states that if the world champion in some game is a Pakistani, then he must also be the champion in Pakistan, while property β states that if some Pakistani (champion) is a world champion, then all champions of Pakistan must be champions of the world" [2, p. 17]. α is equivalent to IP*, and it is obvious that *IP is independent of IP*. The following examples show that *IP and β are independent.

EXAMPLE A. *IP and not β,

$$S = \{x_1, x_2, x_3\} \qquad C(\{x_i\}) = \{x_i\} \qquad i = 1, 2, 3,$$
$$C(\{x_1, x_2\}) = \{x_1, x_2\},$$
$$C(\{x_1, x_3\}) = \{x_1, x_3\},$$
$$C(\{x_2, x_3\}) = \{x_2, x_3\},$$
$$C(\{x_1, x_2, x_3\}) = \{x_1, x_2\}.$$

Here β is violated as $x_2, x_3 \in C(\{x_2, x_3\})$ and $x_2 \in C(\{x_1, x_2, x_3\})$ but $x_3 \notin C(\{x_1, x_2, x_3\})$.

EXAMPLE B. β and not *IP,

$$S = \{x_1, x_2, x_3\} \qquad C(\{x_i\}) = \{x_i\} \qquad i = 1, 2, 3,$$
$$C(\{x_1, x_2\}) = \{x_1, x_2\} \qquad C(\{x_1, x_2, x_3\}) = \{x_1, x_2\},$$
$$C(\{x_1, x_3\}) = \{x_3\},$$
$$C(\{x_2, x_3\}) = \{x_3\}.$$

But *IP requires that

$$C(C(\{x_1, x_3\}) \cup C(\{x_2, x_3\})) = \{x_3\} \subseteq C(\{x_1, x_2, x_3\}).$$

The following theorem shows that *IP is consistent with P, I, and D. In fact we can employ a stronger condition than D, namely anonymity (A).

A: Let $\sigma(1), \sigma(2),\ldots, \sigma(n)$ be any permutation of the integers 1 to n. Then

$$(\forall V) \quad V \subseteq S \quad C(V, R_1, \ldots, R_n) = C(V, R_{\sigma(1)}, \ldots, R_{\sigma(n)}).$$

THEOREM 2. *There exists a choice function that satisfies* *IP, P, I, and *anonymity* (A).

Let $n(xP_iy)$ be the number of individuals in society for whom the indicated preference holds. We proceed by constructing a choice function that satisfies P, I, and A, and demonstrate that *IP holds as well.

Define $xMy \Leftrightarrow n(xPy) \geq n(yPx)$ and for any set $V \subseteq S$ say that

$$x\overline{M}_Vy \Leftrightarrow \exists z_1, z_2, \dots, z_k \in V \ni xMz_1, \quad z_1Mz_2, \dots, z_{k-1}Mz_k, \quad z_kMy.$$

Note that for V fixed $\overline{M}_V$ is a weak order. Further we have the following facts, the proofs of which are obvious.

LEMMA 1. *If* $T \subseteq S$ *and* $x, y \in T$, $x\overline{M}_Ty \Rightarrow x\overline{M}_Sy$.

LEMMA 2. *If* $x, y \in S_1$ *and* $y, z \in S_2$, $x\overline{M}_{S_1}y$ *and* $y\overline{M}_{S_2}z \Rightarrow x\overline{M}_{S_1\cup S_2}z$.

Now define $M(S) = \{x \mid x\overline{M}_Sy \ \forall y \in S\}$.

Proof. For $\phi \neq V \subseteq S$, let $C(V) = M(V)$. Clearly $C(V)$ satisfies P, I, and A. We need only establish that *IP is satisfied too. Assume that $x \in C(\bigcup_i C(V_i))$ and $x \notin C(V)$ where the V_i are such that $\bigcup_i V_i = V$.

Then $\exists y \in S \ni \sim x\overline{M}_Vy$; hence $\nexists z_1, z_2, \dots, z_k \in V \ni xMz_1, \dots, z_kMy$. If $y \in \bigcup_i C(V_i)$, then since $x \in C(\bigcup_i C(V_i))$, $x\overline{M}_{\bigcup_i C(V_i)}y$, we have $x\overline{M}_Vy$ by Lemma 1. Thus $y \notin \bigcup_i C(V_i)$. Assume without loss of generality that $y \in V_1$. For all $z \in C(V_1)$, $z\overline{M}_{V_1}y$ we have $z\overline{M}_Vy$ by Lemma 1. Similarly $x\overline{M}_{\bigcup_i C(V_i)}z$ yields $x\overline{M}_Vz$ so that since $\overline{M}_V$ is a weak order on V, $x\overline{M}_Vy$, which contradicts the original assertion. Q.E.D.

This theorem indicates not only that *IP is consistent with I, P, and A (and therefore of course D), but that a relatively attractive choice procedure based on an extension of majority rule to more than two-element sets is available. It may easily be shown that $C(\ )$ as defined satisfies the Condorcet condition (C), neutrality (N), and a naturally defined monotonicity axiom as well.

C: $(\forall V) \quad V \subseteq S: \qquad x \in V: [\forall y \in V, n(xPy) > n(yPx)] \Rightarrow C(V) = \{x\}$,

N: $\forall x, y, z, w \in S: [(\forall i: xR_iy \Leftrightarrow zR_i'w)$

and

$$\forall i: yR_ix \Leftrightarrow wR_i'z)] \Rightarrow [(x \in C(\{x, y\}, R_1, \dots, R_n) \Leftrightarrow z \in C(\{z, w\}, R_1', \dots, R_n'))$$

and

$$(y \in C(\{x, y\}, R_1, R_2, \dots, R_n) \Leftrightarrow w \in C(\{z, w\}, R_1', \dots, R_n'))].$$

Thus there actually exist normatively compelling choice procedures that satisfy what we would argue is the only technical requirement that should be imposed on a choice procedure: the independence of irrelevant alternatives (see Plott's argument for this remark [1]).

III. CHOOSING CYCLES AND AN IMPOSSIBILITY THEOREM

We now establish some properties of choice functions which satisfy *IP.

THEOREM 3. *Let* $V \subseteq S$ *with* $|V| = m$. *If* $C(\{x_i, x_{i+1}\}) = \{x_i\}$ $i = 1, \dots, m-1$, *then* *IP *implies that*

$$x_1 \in C(V).$$

Proof. The theorem is true for $m = 2$, so suppose it holds for $m = k$. Then

$$x_1 \in C(\{x_1, x_2, \dots, x_k\}) = C\left(\bigcup_{i=1}^{k} C(\{x_i, x_{i+1}\})\right)$$
$$\subseteq C(\{x_1, x_2, \dots, x_{k+1}\}). \qquad \text{Q.E.D}$$

Note that in the above only $m-1$ of the $\binom{m}{2}$ two-element choices were specified so that the theorem remains true under the additional condition $C(\{x_m, x_1\}) = \{x_m\}$. In this case each element of V satisfies the conditions of Theorem 3, so we immediately obtain

COROLLARY 1. *Let* $V \subseteq S$ *be such that the elements of* V *form a cycle. Then* *IP *implies that*

$$C(V) = V.$$

COROLLARY 2. *Let* $V \subseteq S$ *be such that the elements of* V *form a cycle. If*

$$\forall x \in S - V, \qquad \exists T \subseteq S \qquad \text{such that} \quad C(\{x\} \cup T) \subseteq V,$$

then *IP *implies*

$$V \subseteq C(S).$$

Proof. The conditions imply that

$$V = \bigcup_i C(S_i) \qquad \text{where} \quad \bigcup_i S_i = S. \qquad \text{Q.E.D.}$$

From the preceding results we see that when there is a cycle over the entire set of possible social states or a cycle at the "top," so to speak, then *IP requires that the entire cycle be included in the set of chosen alternatives (Schwartz, [11] has established a similar result). It should be pointed out that the choice set may well include other elements in addition. In fact, since $C(V) = V$ satisfies all the path independent type axioms, these axioms in general need to be combined with other axioms in order to ensure that the choice functions obtained are not trivial. Axioms which specify conditions under which certain elements should not be chosen may, however, lead to impossibility theorems.

The choice procedure proposed in Theorem 2 is a natural extension of majority rule in the sense that on two-element sets the procedure simply chooses the majority winner. While this procedure satisfies the usual binary Pareto axiom found in Arrow and Sen and elsewhere, it can easily be shown that it does not satisfy the following Pareto axiom

$$\bar{P}: (\forall V) \quad V \subseteq S \quad xP_i y \qquad \forall i \Rightarrow [x \in V \Rightarrow y \notin C(V)].$$

The following example suffices to demonstrate this. Assume there are three voters and four alternatives and that the preference orders are as follows:

Voter	1	2	3
	x	y	z
	y	z	w
	z	w	x
	w	x	y

It is easily seen that $M(\{x, y, z, w\}) \quad \{x, y, z, w\}$ and since w is Pareto dominated by z, this violates $\bar{P}$.

This example is somewhat disturbing since it shows that the very natural concept of Pareto optimality that is inherited from classical welfare economics—that undominated elements never be chosen—is not satisfied by this seemingly natural extension of majority rule. Thus we see that $M(S)$ can satisfy binary Pareto and not $\bar{P}$. The following theorem indicates that this cannot happen with rules satisfying IP*.

THEOREM 4. *If C satisfies* IP* *then* $P \Leftrightarrow \bar{P}$.

Proof. Clearly $\bar{P} \Rightarrow P$. Now assume that $\bar{P}$ is not satisfied. Then $\exists V \subseteq S$ and $x, y \in V$ such that $xP_i y \; \forall i$, and $y \in C(V)$. By IP*

$$C(V) \subseteq C(C(V - \{x, y\}) \cup C(\{x, y\})).$$

But $y \notin C(\{x, y\})$ and $y \notin C(V - \{x, y\})$. Q.E.D.

The example above indicates, however, that *IP is *not* sufficient to establish this equivalence.

The question arises whether there is *any* extension of majority rule that satisfies $\bar{P}$. Surprisingly, the answer to this question is no. In fact to obtain an impossibility theorem we need only insist that on two-element sets the choice procedure reduces to simple majority rule. We may formulate this requirement as follows:

$$\text{SM}: n(xPy) > n(yPx) \Rightarrow \{x\} = C(\{x, y\}) .$$

THEOREM 5. *If $n \geq 5$ and $|S| \geq 4$, there is no choice function which satisfies* SM, *IP, *and* $\bar{P}$.

Proof. If n is the number of people in society we may write $n = 3m + p$ where $p = 0, 1$, or 2. Thus we may divide the society into at most five groups, N_1, N_2, N_3, N_4, and N_5, with the following preferences in such a way that

$$|N_1| = |N_2| = |N_3| = m \text{ and } |N_4| = 1 \text{ iff } p = 1 \text{ or } 2$$
$$\text{and } |N_5| = 1 \text{ iff } p = 2.$$

N_1	N_2	N_3	N_4	N_5
x	y	z	x	y
y	z	a	y	z
z	a	x	z	a
a	x	y	a	x

Now

$$(\text{SM and *IP}) \Rightarrow \{a\} \subseteq C(C(\{x, y\}) \cup C(\{a\})) \subseteq C(\{a, x, y\}),$$

since $C(\{x, y\}) = \{x\}$ and $a \in C(\{a, x\})$ for any society with the above preferences. Similarly we have

$$\{x\} \subseteq C(C(\{a, y\}) \cup C(\{x\})) \subseteq C(\{a, x, y\}),$$
$$\{y\} \subseteq C(C(\{a, x\}) \cup C(\{y\})) \subseteq C(\{a, x, y\}).$$

Finally, we have

$$a \in C[C(\{a, x, y\}) \cup C(\{y, z\})] \subseteq C(\{a, x, y, z\}).$$

But $\forall i \; zP_i a$ and so $\bar{P}$ is violated. Q.E.D.

The implications of this theorem are quite sweeping. First, the Borda rule, which satisfies SM and $\bar{\text{P}}$ must violate *IP. Second, since Condorcet $\Rightarrow$ SM we have the following consequence to Theorem 5:

COROLLARY. *If* $n \geq 5$ *and* $|S| \geq 4$, *there is no choice function that satisfies* C, *IP, *and* $\bar{\text{P}}$.

This establishes a basic incompatibility between the Condorcet principle which to many people is a deeply attractive property of a choice procedure and other appealing ethical requirements on social choice: $\bar{\text{P}}$ and *IP. To some Condorcet or SM may seem too stringent as ethical requirements on social choice. These individuals might be satisfied with a choice procedure that reduced to some egalitarian form on two-element sets, e.g., representative majority rule. Thus they might say that Theorem 5 is not a "telling" argument against the possibility of social choice being democratic and satisfying the general Pareto principle.

In our view a principal insight of social choice theory is that if a choice function is generally required to satisfy a rationality requirement, then it is essentially oligarchic. Further, within the class of oligarchic rules the only procedures which may legitimately be called democratic are unanimity-type rules. Thus as a practical matter for democratic choice procedures there must be preference configurations which can lead to cycles. It turns out that this condition and the neutrality axiom are sufficient for *IP and a version of the Pareto principle to be in conflict. In the following we use a strengthening of $\bar{\text{P}}$.

$$\overline{\text{SP}}: \quad \forall i \quad xR_iy \qquad \text{and} \qquad \exists i \quad xP_iy \Rightarrow [x \in V \Rightarrow y \notin C(V)].$$

THEOREM 6. *Let C be a choice function defined on S satisfying neutrality* (N). *Assume there is a set* $V = \{v_1, v_2, \ldots, v_k\} \subset S$ *and a preference configuration* $\Pi = (R_1, R_2, \ldots, R_n)$ *such that*

$$\begin{aligned}
\{v_1\} &= C(\{v_1, v_2\}, \Pi),\\
\{v_2\} &= C(\{v_2, v_3\}, \Pi),\\
&\vdots\\
\{v_{k-1}\} &= C(\{v_{k-1}, v_k\}, \Pi),\\
\{v_k\} &= C(\{v_k, v_1\}, \Pi).
\end{aligned}$$

Then C cannot satisfy *IP *and* $\overline{\text{SP}}$.

Proof. Consider a new preference configuration $\Pi' = (R_1', \ldots, R_n')$ satisfying the following conditions.

(1) $\forall x, y \in V \quad xR_iy \Leftrightarrow xR_i'y \quad i = 1, \ldots, n.$

(2) For some $y \in S - V$,

$$\begin{aligned} \forall i \qquad & v_1 I_i v_2 \Rightarrow v_1 I_i' y, \\ & v_1 P_i v_2 \Rightarrow v_1 P_i' y \text{ and } y P_i' v_2, \\ & v_2 P_i v_1 \Rightarrow v_1 P_i' y. \end{aligned}$$

If C satisfies $\overline{\mathrm{SP}}$ and N, then

$$C(\{v_1, y\}, \Pi') = \{v_1\},$$

and N implies that

$$C(\{y, v_2\}, \Pi') = \{y\}.$$

Thus by Corollary 1 to Theorem 3, *IP and N require

$$C(V \cup \{y\}, \Pi') = V \cup \{y\}.$$

Q.E.D.

This result shows the difficulty of finding an "acceptable" nonunanimous democratic choice rule. If any democratic choice procedure satisfies *IP then it cannot satisfy $\overline{\mathrm{SP}}$. In light of Arrow's rather convincing justification of *IP we can see that our deeply held feelings about democratic choice and Pareto optimal choice are incompatible.

IV. DISCUSSION

It seems to us that path independent choice is desirable for two basic reasons. First, while we may have institutional procedures that work very naturally and satisfactorily on two-element sets, we may not be so satisfied with various methods of extending these procedures to choice over larger sets. If, however, an extension of choice of two-element sets satisfies *IP, at least we may be sure that decomposing the choice problem in order to employ our appealing procedure on two-element sets repeatedly will not lead us to a clearly unsatisfactory position. Secondly, even in some democratic institutions a small group of individuals may be able to alter the agenda (the way that choice over V is broken up for consideration). *IP guarantees that while they may manipulate the agenda to their advantage, they cannot force the society to an unsatisfactory choice. But while *IP seems to be an attractive normative requirement, accepting it forces us to choose between a very reasonable version of the Pareto principle and a requirement that social choice be democratic in two-element choice. Thus while abandoning the very weakly justified principle of IP* allows us to escape from the Arrow-type impossibility theorems, a new set of impossibilities have appeared. These results are more disturbing since they force us to recognize an incompatibility between principles that, unlike the rationality requirements, have strong normative justification.

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A Note on "Fulfilled Expectations" Equilibria

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1. INTRODUCTION AND SUMMARY

In this note, the "fulfilled expectations" equilibrium notion, found in Green [3], Grossman [5], and Kihlstrom and Mirman [8], is used to analyze a simple model of a repeated futures market. (The model used is similar to one advanced by Stiglitz [11].) We are primarily interested in examining this equilibrium notion in the context of this simple model, and we find that it gives rise to a number of very interesting economic phenomena.

The model is specified in Section 2. The market considered exists solely for insurance purposes: Speculators have random nonmarket income which may be statistically dependent on a price which will prevail in an upcoming spot market. By speculating in a futures market for the spot market commodity, they can hedge against adverse fluctuations in their nonmarket income. (Presumably, this takes place in a world without a complete set of contingent markets.)

In Section 3, the "fulfilled expectations" equilibrium notion is advanced in the context of this model. Economic motivation for this notion is given. An economic difficulty is immediately encountered: Information originally possessed by no speculator may be transmitted by equilibrium prices. This is easily resolved, by restricting attention to prices which contain no more information than is possessed by all the speculators. However, when attention is so restricted, another problem is encountered: An equilibrium need not exist. An example of this is given in Section 4.

Another difficulty with this equilibrium concept arises in connection with the standard notion of a demand correspondence. Roughly, a

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speculator, upon submitting his demand correspondence to a Walrasian auctioneer, may not be willing to let the auctioneer select a point in the correspondence so as to balance supply and demand. And the auctioneer, in balancing supply and demand in this fashion, may reveal information not revealed by prices. In the context of this model this problem is easily avoided (by assuming strictly concave utility). But this does illustrate an incompatibility between the usual concept of a demand correspondence and "fulfilled expectations" equilibria. All this is discussed in Section 5.

In Section 6, the role of the market in insuring speculators is considered. The activity which takes place in the futures market is pure gambling about a future spot price. One might therefore expect (following [10]) that, given independence between the spot price and investors' nonmarket income, rational risk averse speculators will not take part in the market. This is verified under very stringent independence assumptions. But it does not suffice simply to have independence between spot price and each speculator's nonmarket income (either unconditionally or conditional on the information the speculator brings to the futures market). A speculator may gain information from futures prices which causes a conditional dependence between his nonmarket income and the spot price, allowing him to insure himself. In Section 7, an example which illustrates this is given. In addition, this is an example with two equilibria, one of which is Pareto-superior to the other. (Because this is a partial equilibrium analysis, this statement should be regarded with some suspicion.) And this is an example where a single speculator wishes to reveal, by nonmarket means if necessary, information which he possesses.

2. FORMULATION

Imagine a futures market for, say, wheat, which meets once a year, midway through the season. All contracts made in the futures market are for delivery at the end of the season and must be covered at the then prevailing spot price. This spot price, denoted q, is random, depending perhaps on the unknown size of the crop and the supply of supplementary foodstuffs. Since the net position taken in the futures market is zero (for every contract purchased, one must be sold), it is assumed that the spot price does not depend in a causative sense on the futures market transactions.

There is a finite set of speculators who may participate in this market, indexed by $i = 1,..., n$. Each comes to the market with some information concerning the upcoming spot price; the information possessed by i is denoted s_i. Each speculator is also assumed to have a (random) income derived from sources other than speculation in the futures market; i's nonmarket income is denoted y_i. These nonmarket incomes may be statistically dependent on the spot price. (The obvious example is the farmer-cum-speculator, who will sell his crop on the spot market.) Each speculator attempts to maximize the expectation of his utility of his total income, conditioned on the information he possesses, including information that can be gleaned from the futures market price. That is, if p denotes the futures market price and i takes a net long position of x in the futures market, his total income is $x(q-p)+y_i$. His utility function is denoted U_i, and we assume he chooses x so as to maximize the expectation of $U_i(x(q-p)+y_i)$, conditional on his information.

Moreover, we imagine that this takes place year after year and every year is an independent and identical copy of every other year. Implicit in this are unrealistic behaviorial assumptions (beyond the lack of realism in i.i.d. crops): Speculators do not carry over wealth from one year to the next. Instead, it is as though consumption in any year must exactly equal the net income from the year before. Also, it is assumed that speculators are price takers and do not collude.

Formally, there is an underlying probability space $(\Omega, \mathscr{F}, P)$, on which are defined random variables y_i $(i=1,\ldots,n)$, s_i $(i=1,\ldots,n)$, and q. A generic $\omega \in \Omega$ is a "state of the world" for the year in question. (Successive years would be represented by a sequence $(\omega_0, \omega_1, \ldots)$ drawn from $\Omega^{\{0,1,\ldots\}}$, endowed with the standard product σ-field and probability.) Also specified in the model are the utility functions U_i $(i=1,\ldots,n)$.

The following mathematical assumptions are made throughout. (For tractability, Assumptions 2 and 3 will be violated in the example of Section 4.)

Assumption 1. Each $U_i\colon(-\infty,\infty)\to(-\infty,\infty)$ and is concave and strictly increasing.

Assumption 2. There exists scalar Q such that $0\leqslant q\leqslant Q$. (The spot price is always positive, and is bounded above by, say, $\$10^{100}$.)

Assumption 3. For each i, there exists scalar Y_i such that $|y_i|\leqslant Y_i$.

Assumption 4. $(\Omega,\mathscr{F})$ is a Borel space. That is, there exists a one-to-one map from Ω to R (the real line) such that the map and its inverse are measurable with respect to $\mathscr{F}$ and the Borel field on R. Note that Borel subsets of R^m, $m=1,\ldots,\infty$, are Borel, so this is not much of a restriction. (See Breiman [2, Sect. 5.3].)

3. FULFILLED EXPECTATIONS EQUILIBRIA

Depending on the state of nature ω, speculators will possess various pieces of information, thus their demands at the futures market will depend on ω. A single price cannot therefore be expected to balance demands in all states of nature; an equilibrium price must be a function of ω. Formally, a *price scheme* is a nonnegative real valued (Borel) measurable function on $(\Omega, \mathscr{F})$. (Because $0 \leq q \leq Q$, we further assume that $0 \leq p \leq Q$; it is unlikely that one would ever observe a futures price which is greater or less than any possible value of the spot price.) Such a price scheme p prescribes to each state of nature ω a price, $p(\omega)$, in the futures market.

Following [3, 5, 8], the following condition is required for a price scheme p to be a *fulfilled expectations equilibrium* (or simply, an equilibrium):

$$0 \in \sum_{i=1}^{n} \arg\max_{x} E[U_i(x(q-p)+y_i) \mid s_i, p]. \tag{1}$$

To be mathematically precise, it is necessary to fix a regular version of conditional probability for each p and s_i and define conditional expectations from them. The assumption that $(\Omega, \mathscr{F})$ is Borel is employed here, cf. [2, Sect. 4.3]. It is straightforward to show that the satisfaction of (1) is independent of the version of regular conditional probability chosen. Other mathematical facts and conventions concerning (1) that should be noted: Fix a price scheme p, and define

$$u_i(x, \omega) = E[U_i(x(q-p)+y_i) \mid s_i, p](\omega)$$

for $x \in (-\infty, \infty)$. From the boundedness assumptions, $u_i(x, \omega)$, is well defined, finite, jointly measurable in (x, ω), and continuous and concave in x. By concavity, $u_i(\cdot, \omega)$ can be continuously extended to $x = \pm\infty$. Throughout, by $\arg\max_x E[U_i(x(q-p)+y_i) \mid s_i, p](\omega)$ we mean the set of x from $[-\infty, \infty]$ which achieves the maximum in $u_i(\cdot, \omega)$. The sum in (1) is then defined as the usual addition of sets, with the following (economically reasonable) convention: If for some ω, $\infty \in \arg\max u_i(\omega, x)$ for one i and $-\infty \in \arg\max u_i(x, \omega)$ for some other i', then the sum in (1) is $[-\infty, \infty]$.

The economic motivation for (1) runs as follows. Suppose the price scheme p is "used" year after year. Eventually, every speculator (being a very good statistician) learns the function $p(\omega)$. (Actually, it suffices for him to learn the joint distribution of (p, s_i, q, y_i).) So in the long run, speculators will use $p(\omega)$ to infer what they can about the state of nature. Then

$\arg\max_j E[\quad s_i, p]$ is the demand correspondence of speculator i, and (1) becomes the usual market clearing condition.

In this rationale, "eventually" and "in the long run" deserve to be regarded with skepticism. One might imagine a dynamic, temporary equilibrium process where speculators formulate Bayesian (or some other estimates) of p and update these each year, as more data come in. And one might envision a proof of convergence in this process to a price scheme for which (1) holds. But such an argument is far beyond the scope of this paper. Or one might imagine this process initiated with a scheme p (for which (1) holds) as one point priors for all speculators. Then the speculators would find their expectations being fulfilled.

A parable of a Walrasian auctioneer can also be constructed to rationalize (1). The auctioneer in question is omniscient, knowing ω at the time of the futures market. He uses a price scheme p: knowing ω, he calls out $p(\omega)$. Further, the local version of the S.E.C. has forced him to publish the function p that he is using. So when he calls out $p(\omega)$, the speculators will use this to infer what they can about ω. Then (1) is again the market clearing condition. (At this point, the reader is entitled to conclude that (1) is a highly idealized, and highly suspect, way of modeling the "prices transmit information" phenomenon.)

An immediate difficulty with (1) concerns the amount of information allowed to be contained in a price scheme p. Unless one really believes in omniscient auctioneers, it seems reasonable to restrict attention to price schemes p which contain no more information than is possessed by all the speculators taken together; i.e., which are $(s_1, \ldots, s_n)$-measurable. But one price scheme which (trivially) satisfies (1) is $p - q$, the price in the futures market identically the spot price. In general, this p is not $(s_1, \ldots, s_n)$-measurable. So we must add to (1) the condition that p is $(s_1, \ldots, s_n)$-measurable. In the sequel, this will be implicit whenever we say that p is an equilibrium price scheme.

4. AN EXAMPLE OF NONEXISTENCE

One difficulty with the equilibrium notion of Section 3 is that equilibria need not exist. We consider a model with two groups of investors, the "informed" and the "uninformed." Following the sort of argument given in [3, 5, 8], it is seen that in this example, if equilibrium prices exist, they reveal the superior information of the informed to the uninformed. But, for the given data, if both classes possess exactly the information of the informed, then the unique equilibrium prices do not reveal this information. So no equilibrium can exist.

Specifically, there are equal numbers of informed and uninformed speculators. For simplicity, we assume there is one of each. Each has constant absolute risk aversion; $U_i(z) = -\exp(-z)$ for $i = 1, 2$. The possible states of the world are partitioned into two sets, Ω_1 and Ω_2. The probabilities of Ω_1 and Ω_2 are each $\frac{1}{2}$. Conditional on $\omega \in \Omega_1$, q is Normally distributed with mean m_1 and variance σ^2, and y_2, the nonmarket income of the second speculator, is $k_1 q$, for nonnegative constant k_1. Conditional on $\omega \in \Omega_2$, q is Normal with mean m_2 and variance σ^2, and y_2 is $k_2 q$. The nonmarket income of speculator 1 is constant, thus can be ignored.

Speculator 1 receives information s_1 defined by $s_1(\omega) = 1$ if $\omega \in \Omega_1$ and $s_1(\omega) = 2$ if $\omega \in \Omega_2$. Speculator 2 is given no information (s_2 constant). (Thus speculator 2's prior on q is an even mixture of two Normal laws, which in general is *not* Normal.) Therefore, if p is an (s_1, s_2)-measurable price scheme, p is s_1-measurable and reveals nothing to speculator 1. Then given $s_1 = s$ ($s = 1, 2$), at price p, speculator 1 will demand

$$x_1(p, s) = \arg\max_x E[-\exp\{-x(q - p)\} \mid s_1 = s] = (m_s - p)/\sigma^2.$$

Assuming $m_1 \neq m_2$, if p is an equilibrium price scheme, p must reveal s_1 to speculator 2. For speculator 2 knows, given p, how much he is demanding. Since the sum of his demand and that of speculator 1 must be zero, he knows how much speculator 1 demands. Thus he can invert $x_1(p, s)$ to find s.

Therefore, if p is an equilibrium price scheme, speculator 2's demand is calculated with knowledge of s_1. If $s_1 = s$ ($s = 1, 2$) and the price is p, he demands

$$\begin{aligned} x_2(p, s) &= \arg\max_x E[-\exp\{-(x(q - p) + k_s q)\} \mid s_1 = s] \\ &= ((m_s - p)/\sigma^2) - k_s. \end{aligned}$$

Consider the data $m_1 = 4$, $m_2 = 5$, $k_1 = 2$, $k_2 = 4$, and $\sigma^2 = 1$. If $(p(1), p(2))$ is an equilibrium price scheme, it must be that $x_1(p(s), s) + x_2(p(s), s) = 0$ for $s = 1, 2$. This is equivalent to $p(s) = m_s - k_s/2$, or, for the given data, $p(1) = p(2) = 3$. But these prices *do not* reveal s_1 to speculator 2. Hence, there does not exist an equilibrium price scheme.

Note that this example of nonexistence of equilibria is not stable under small perturbations of the data. If the data are changed slightly (except in certain, systematic ways), an equilibrium price scheme will appear. Compare this with the more complicated example of nonexistence given by [4], which is stable under all small perturbations in a very strong topology (however, not in the weak topology of probability distributions).

Jordan [7] gives a general result about nonexistence, in a slightly different context. Also, compare this example with one given by Hart [6]. In both, the difficulty is that continuous changes in prices can lead to discontinuous changes in demand – here because of a discontinuity in information and in Hart's example because of a discontinuity in the feasible set.

5. INFORMATION FROM QUANTITY

Another defect with (1) is that, in the context of repeated markets, the standard notion of a demand correspondence is defective. Consider the usual parable of the Walrasian auctioneer. The auctioneer calls out a price, and each agent submits a set of budget feasible net demands among which the agent is indifferent. The auctioneer's job is to choose from each agent's submission one net demand, such that the sum of net demands is (less than) zero. Because each agent is indifferent between elements in his correspondence, he (presumably) is willing to let the auctioneer make this final choice. But in the context of a repeated market, the agent may not be willing to give the auctioneer this freedom. And if speculator i does allow the auctioneer this freedom, the auctioneer's choice may contain information not contained in either s_i or p. An example will illustrate how this can happen.

Consider a model with four states of nature, (10), (11), (30), and (31), each occurring with probability $\frac{1}{4}$. There are two speculators (or two groups) who are identical except that the first possesses more information. In state (ij), $q = i$ and $y_1 = y_2 = j$. Each speculator has a utility function which is strictly concave in $(-\infty, -\frac{1}{2})$ and $(\frac{3}{2}, \infty)$, is linear in $(-\frac{1}{2}, \frac{1}{2})$, and is linear with smaller slope in $(\frac{1}{2}, \frac{3}{2})$.

Speculator 1 receives information $s_1 = 1$ if the state is (10) or (31) and $s_1 = 2$ if the state is (11) or (30). Speculator 2 receives no information (s_2 constant).

We claim that $p = 2$ is an equilibrium price scheme. At this price, if $s_1 = 1$ and speculator 1 demands x, he gets expected utility $\frac{1}{2}\{U(-x) + U(x+1)\}$, so he optimally demands $x_1(1) = -\frac{1}{2}$. And if $s_1 = 2$ and he demands x, his expected utility is $\frac{1}{2}\{U(-x+1) + U(x)\}$, so he optimally demands $x = \frac{1}{2}$. Speculator 2 has no information, so if he demands x, his expected utility is $\frac{1}{4}\{U(-x) + U(x) + U(1-x) + U(1+x)\}$, which is maximized by any $x \in [-\frac{1}{2}, \frac{1}{2}]$. So, by (1), this is an equilibrium price scheme.

But if a Walrasian auctioneer is running this market, he must assign to speculator 2 $\frac{1}{2}$ contract if $s_1 = 1$ and $-\frac{1}{2}$ contract if $s_1 = 2$, in order to equate supply and demand. If speculator 2 does not quickly realize that there is a dependence between his nonmarket income, the spot price, and the number of contracts the auctioneer assigns him, he will eventually be disappointed to learn that half the time his entire income is $\frac{3}{2}$ (in states (11) and (31)) and half the time it is $-\frac{1}{2}$, which yields less expected utility than he anticipated.

What is happening here should be apparent. For a price scheme p which satisfies (1), the auctioneer must select functions x_i^* on Ω $(i = 1, \dots, n)$ such that $x_i^* \in \arg\max E[- \mid s_i, p]$ for each i and $\sum_{i=1}^{n} x_i^* = 0$, P-a.s., where, of course, $x_i^*(\omega)$ is to be the number of contracts allotted to i if the state is ω. There is no problem in doing so; in fact, a standard measurable selection argument can be constructed to obtain functions x_i^* which are $(s_1, \dots, s_n)$-measurable. (Since everything is Borel, use von Neumann's selection theorem; cf. [1].) But, in general, the selection cannot be accomplished so that each x_i^* is (s_i, p)-measurable, witness the example. So x_i^* (which certainly must be known to speculator i) may transmit information not contained in (s_i, p). And the demand correspondence is computed on the assumption that the allotted number of contracts is (s_i, p)-measurable. For x_i^* which are not (s_i, p)-measurable, the speculator may be unpleasantly surprised. (Pleasant surprises are also possible; the example can easily be modified so that this is the case.)

Two possible modifications of the equilibrium notion can be suggested to resolve this difficulty. The first is to require that there exist functions x_i^*, (s_i, p)-measurable, respectively, such that $x_i^* \in \arg\max_x E[- \mid s_i, p]$ for each i and $\sum_{i=1}^{n} x_i^* = 0$, P-a.s. The second (and certainly, the more provocative) is to require that there exist functions s_i' and x_i^* for each i such that x_i^* is (s_i, s_i', p)-measurable and $x_i^* \in \arg\max_x E[- \mid s_i, s_i', p]$ for each i and $\sum_{i=1}^{n} x_i^* = 0$, P-a.s. The s_i' are to be interpreted as private "quantity" signals (perhaps being sent out by the auctioneer). If no other quantity information besides x_i^* is present, $s_i' = x_i^*$ seems appropriate.

Note that this problem will not be encountered if the U_i are strictly concave. (Note also that in the example, it is the utility function of speculator 2 which must have "flat spots." The example remains unchanged if for U_1 we take any utility function, so long as it is strictly concave at the argument $\frac{1}{2}$.) For if the U_i are strictly concave, then either $\arg\max_x E[- \mid s_i, p]$ is a singleton or $p = q$, $P[\cdot \mid s_i, p]$-a.s. (in which case, the $\arg\max_x = [-\infty, \infty]$). Thus suppose p satisfies (1). Select measurable x_i^* such that $x_i^* \in \arg\max_x E[- \mid s_i, p]$ for each i and $\sum_{i=1}^{n} x_i^* = 0$, P-a.s. (Denote the P-null set where this fails by N.) Then for each i and $\omega \notin N$, either $x_i^*(\omega') = x_i^*(\omega)$ for all $\omega' \notin N$ such that $s_i(\omega') = s_i(\omega)$ and $p(\omega') = p(\omega)$, or $P[p = q \mid s_i, p](\omega) = 1$. For those ω where the latter holds, the extra information is contained in $x_i^*(\omega)$ is useless, because the speculator "knows" ($P[\cdot \mid s_i, p](\omega)$-a.s.) that $q = p$. And in either case,

$E[U_i(x_i^*(p)(q-p)+y_i) \mid s_i, p] = \max_x E[\cdot \mid s_i, p]$, P-a.s.; i.e., the speculator "gets" precisely the expected utility that he anticipates. (The rigorous proof of this last statement is left to the reader.)

Because of the simplicity of the model examined here, we will use the strict concavity assumption in the following section. Any judgment as to the merits of the two suggested repairs of the equilibrium notion must await study in a more robust setting.

6. INSURANCE MOTIVES AND PURE GAMBLING MARKETS

Suppose the example in the Section 4 was modified so that each speculator received the signal s_1. Then $p = 3$ would be an equilibrium. The economic role played by this market is apparent: speculator 2 finds that his nonmarket income is positively correlated with the spot price, so he takes a net short position in the futures market. In order to obtain this insurance, he pays a premium to speculator 1: he sells for $3 a contract which he will have to "buy back" at an expected price of $4 or $5 (depending on s_1). This premium is the incentive to speculator 1 to take part.

This is roughly the classic Keynes–Hicks theory of futures markets: a wheat farmer's nonmarket income is positively correlated with the spot price at the harvest, being the spot price times the size of his crop. He therefore takes a short position in the futures market, "selling" a part of his crop at a certain price and insuring himself against a fall in prices. Similarly, agents with incomes negatively correlated with the spot price, General Mills, say, take long positions for insurance motives. (Another rationale for futures markets is that they provide signals that allow for efficient allocation. Grossman [5] hypothesizes that the futures markets act in this fashion, by rewarding the information gathering activities of speculators. The model used here is clearly insufficient for viewing such phenomena.)

Except for this insurance phenomenon, the model is one of a pure gambling market, in that speculators are simply gambling about q. Following Stiglitz [10], one expects that any fulfilled expectations equilibrium in a pure gambling market must entail no risk averse investor taking a nonzero position. So it is natural to ask what conditions on the s_i, y_i, and q would lead to the trivial sort of equilibrium that a pure gambling market must entail.

Perhaps surprisingly, it is not sufficient to have that, for each i, q and y_i are independent and/or s_i-conditionally independent. An example where this is the case and where there is a nontrivial equilibrium is given in the

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following section. What is happening in this example is that while one speculator's y_i and q are (s_i-conditionally) independent, it is not true that y_i and q are $(s_1, \ldots, s_n)$-conditionally independent. The equilibrium price scheme p reveals other speculator's information to this speculator, causing a conditional dependence which he can exploit for insurance purposes.

Sufficient conditions for Stiglitz' result are quite stringent. Assume that the U_i are strictly concave, so that if p is an equilibrium price scheme, ($(s_1, \ldots, s_n)$-measurable) functions x_i^* can be selected so that $x_i^* \in \arg\max_x E[\cdot \mid s_i, p]$ and $\sum_{i=1}^{n} x_i^* = 0$, P-a.s. If for each i, y_i and the σ-field generated by $(q, s_i, \ldots, s_n)$ are s_i-conditionally independent, then for any equilibrium price scheme p, $0 \in \arg\max_x E[\text{—} \mid s_i, p]$, P-a.s., for each i. (One additional technical assumption will be required in the proof: For the scheme p under consideration, it must be possible to produce integrable x_i^* as above.)

That is, each speculator will at best be indifferent between participating in the market and not doing so, and if he is at all unsure about q (given p and s_i), he does not want to participate. The condition, that y_i and $\mathscr{F}(q, s_1, \ldots, s_n)$ are s_i-conditionally independent, roughly means that each speculator possesses as good information about his own income as he would if he knew q and the signals being received by all the agents. This is true, say, if each y_i is constant, if y_i is a function of s_i, or if the y_i are independent of $(q, s_1, \ldots, s_n)$.

A sketch of the proof of this follows: Let N be the P-null set on which $x_i^* \in \arg\max_x E[\cdot \mid s_i, p]$ or $\sum_{i=1}^{n} x_i^* = 0$ or $E[U_i(x_i^*(q - p) + y_i) \mid s_i, p] = \max_x E[\cdot \mid s_i, p]$ fails. Then for all $\omega \notin N$ and for all i, $E[x_i^*(q - p) \mid s_i, p] \geqslant 0$. For if $E[x_i^*(q - p) \mid s_i, p] < 0$, then $E[U_i(x_i^*(q - p) + y_i) \mid s_i, p] = E[E[U_i(x_i^*(q - p) + y_i) \mid s_i, p, y_i] \mid s_i, p] \leqslant$ [Jensen's inequality] $E[U_i(E[x_i^*(q - p) \mid s_i, p, y_i] + y_i) \mid s_i, p] =$ [by the independence hypothesis, cf. [9, 25.3.A]] $E[U_i(E[x_i^*(q - p) \mid s_i, p] + y_i) \mid s_i, p] <$ [strict monotonicity of U_i] $E[U_i(y_i) \mid s_i, p]$, which contradicts $E[U_i(x_i^*(q - p) + y_i) \mid s_i, p] = \max_x E[\cdot \mid s_i, p]$. But since $\sum_{i=1}^{n} x_i^* = 0$, P-a.s., $\sum_{i=1}^{n} E[x_i^*(q - p)] = 0$. Thus for each i, $E[x_i^*(q - p) \mid s_i, p] = 0$, P-a.s. Then repeating the Jensen's inequality–conditional independence argument yields $E[U_i(x_i^*(q - p) + y_i) \mid s_i, p] \leqslant E[U_i(y_i) \mid s_i, p]$, P-a.s., which is the desired result.

7. AN EXAMPLE

Consider a model with four states of the world $(1, -1)$, $(1, 0)$, $(2, -1)$, $(2, 0)$, each occurring with probability $\frac{1}{4}$. There are two speculators. Speculator 1 has nonmarket income $y_i = 0$. In state (i, j), $q = i$ and $y_2 = j$.

Speculator 1 receives information s_1 where $s_1 = 1$ if $(1, -1)$ or $(2, 0)$ and $s_1 = 2$ if $(1, 0)$ or $(2, -1)$. Speculator 2 receives no information. So each speculator's nonmarket income is both independent and conditionally independent given his own information (not including p) of q. Each speculator has utility function $U(z) = z - z^2$ in the range $(-\infty, 0.4)$, strictly concavely extended (in arbitrary fashion) to $(0.4, \infty)$.

It is claimed that $p = \frac{4}{3}$ if $s_1 = 1$ and $p = \frac{5}{3}$ if $s_1 = 2$ are equilibrium prices. The calculations are left to the reader. But note that this is reasonable: These prices reveal s_1 to speculator 2. If $s_1 = 1$, then y_2 is positively (conditionally) correlated with q, and speculator 2 will desire to take a net short position. (He sells $\frac{3}{10}$ of a contract.) Speculator 1 is content in buying $\frac{3}{10}$, because the expected spot price, $\frac{3}{2}$, exceeds the futures price, $\frac{4}{3}$. And conversely, if $s_1 = 2$, y_2 and q are negatively (conditionally) correlated, so speculator 2 wishes a net long position, even though the expected spot price is less than the futures price.

It is easy to verify that $p = \frac{3}{2}$ is also an equilibrium price scheme, in which both speculators demand exactly 0. Note that the first equilibrium makes each speculator ex ante (hence, long run average ex post) better off; speculator 2 gets insured and speculator 1 gets the insurance premium. So we have two equilibria, one pareto-superior to the other. Note also that if speculator 2 knows s_1, only the first equilibrium can result. So if, in the original circumstances, the second equilibrium "came about," it would be worthwhile for speculator 1 to reveal s_1 by nonmarket means (even if there were two large groups of each type, so long as the revelation can be achieved at sufficiently small cost). Of course, an increase in informedness need not result in greater welfare. Suppose in this model that speculator 1 knew precisely what state prevailed. Then (holding to the price taking assumption, so that this makes more sense if there are two large groups of each type) the only possible equilibrium is $p = q$, where speculator 2 gets no insurance, and speculator 1 gets no insurance premium.

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Decentralized Control Problems in Neumann-Economies*

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1. LITERARY ANTECEDENTS

Our paper has been born from the cross breeding of three trends in research.

1. *Mathematical modeling of the control mechanism of economic systems.* The first experiments in this direction were made by the neoclassical school in the 19th century by Cournot [7] and Walras [34], then, during the 20th century renaissance of the general equilibrium theory, by Samuelson [29], Arrow and Debreu [9] and their followers. Recently, several authors have attempted to part, partially or entirely, with the neoclassical schemes of reasoning in the mathematical modeling of the control of economic systems. Important pioneers were Hurwicz [13], as well as Marschak and Radner [24].

2. *The dynamic Leontief and the Neumann models.* (See, e.g., [4, 20, 27]). As opposed to the research trend reviewed in paragraph one, these were intended originally to be only models of the *real* sphere and were not linked to the description of *control.* The first steps towards analyzing the control of the Leontief and the Neumann economics were made by Sargan [30], Leontief [21], Lovell [22], and Jorgenson [14].

3. *Application to economics of mathematical control theory and dynamic programming.* Up to now these have been mainly used for the formalization of Keynesian macroeconomics, capital theory, and the theory of growth. (See, e.g., Tustin [31], Lange [19], Dorfman [10],

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* J. Kornai wrote a longer study [18], in which he outlined a few main ideas of the present paper in 1973, during his visit with the Institute of Mathematical Studies in the Social Sciences, Stanford University, California, and with the Institute of Mathematical Economics, Bielefeld University, GFR. Thanks are due to the two institutions for promoting the work. Later on the authors jointly continued the research in Budapest. We are indebted to the Institute of Economics, Hungarian Academy of Sciences for support, and to B. Martos, I. Dancs, Z. Kapitány and an anonymous referee of this Journal for valuable comments.

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Radner [28], and Arrow and Kurz [2].) Also, the control of econometric simultaneous equation systems with the aid of control theory devices is known (see Chow [6]). It has also been used to study some dynamic planning problems, e.g., in the study by Kendrick [15]. But up to now it has been used only exceptionally (e.g., in McFadden [23]) for modeling the control mechanism of economic systems.

The amalgamation of the three trends was attempted in a study written by one of the authors of the present article, J. Kornai, with coauthor B. Martos [16], [17] about autonomous (vegetative) control, and in several other papers improving upon the Kornai–Martos model: Dancs, Hunyadi, and Sivák [8], Virág [33], and Bródy [5]. The present article also belongs to this series. Its aim, like that of the earlier studies, is to "cross breed" the three trends: to foster the researches of the first trend, that is of the abstract modeling of the control mechanisms while using the apparatus and results of the second trend (dynamic Leontief and Neumann models) and of the third (control theory). This crossbreeding involves, of course, not only advantages, but also strong restrictions. We are compelled to use in combination the strong assumptions applied on the one hand in the literature of the Leontief and the Neumann models, and by mathematical control theory, on the other hand.

Having briefly reviewed the literary antecendents of our paper, we are going to review the models and the conclusions that can be drawn with their aid. In some places we will make references to literary sources also under way, and, in a later part of the study a separate section will be devoted to comparing our results with those of other papers.

2. The Model[1]

General Remarks

Exclusively *dynamic systems* will be investigated. Time is an integer variable: $t = 0, 1, 2, \ldots$. The time unit is called a *period*. A stock variable always refers to a state at the beginning of the period (e.g., initial stock), while the flow takes place in the course of the period.

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[1] The main principles of notation: *Matrices* will be denoted by italic capital letters; their elements by the same lowercase letters and double subscript. *Vectors* are denoted by italic lowercase letters, their components by the same letter with a subscript. Transposition is denoted by a prime. A *diagonal matrix* is denoted by the vector in the main diagonal between the symbols $\langle\ \rangle$. The logical product of two matrices (by elements) is denoted by $\otimes$ and their inequality by $\geqq$ and $>$, where the former does not exclude that every pair of elements should be equal, while the latter excludes it. 1 stands for unity as well as for the summation vector of dimension *n*.

The *real sphere* of the economic system comprises the variables of the basic real processes (production, productive consumption, personal and public consumption, turnover) and the relationships existing between them. On the theoretical plane the *control sphere* controlling the real sphere is distinguished from it. In some notations reference is made to the real sphere with an R and to the control sphere by using C.

The economy has n kinds of *products*; they are produced by n *sectors* of the economy.

Two kinds of economies will be examined. The first is called a *Neumann-economy* because, as regards its essential features, it corresponds to an economy having a constant structure, increasing at a steady rate and proceeding around a Neumann-path. The other one will be called an ϵ-asymptotic Neumann-economy, or, for short, an *asymptotic economy*.[2] By simplifying the explanation somewhat we can say the latter is such a dynamic Leontief system, coming very near to a Neumann-economy with a constant structure. A more precise definition of the two economies will emerge from the detailed discussion of the assumptions and theorems. For the most part, the assumptions used in modeling the two economies will be identical. In denoting the alternative assumptions and theorems which differ in the two economies, the distinctions N (Neumann-economy) and A (asymptotic economy) will be applied.

Variables

The variables used in the model are the following.

$r(t)$ = the vector of *production* with n components. The variable r_j is the output of sector j.

$Y(t)$ = the matrix of *purchases*, of the size $n \times n$. The variable y_{ij} is the quantity of the product i purchased by the sector j.

$w(t)$ = the vector of the *output stock* with n components. The variable w_j is the stock accumulated by sector j from its own product in its own inventory.

$V(t)$ = the matrix of the *input stock*, of the size $n \times n$. The variable v_{ij} is the stock of product i accumulated in the warehouse of sector j.

$S(t)$ = the *slack input stock*[3] of the size $n \times n$. The variable s_{ij} is the part of the total input stock v_{ij} above the technologically necessary quantity. This definition will still be reverted to.

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[2] Of course, the first type is a special case of the second one.

[3] The denomination "slack" indicates the part of the resources which is not utilized. The name "slack" is itself a neutral word, it does not involve any value judgement. It contains the *reserves* deliberately built up for balancing random disturbances, as well as the *surpluses*, the residuals unjustified from the point of view of rational management.

The Real Structure

There are two kinds of technological coefficient matrices in the model:

$A(t)$ = the *matrix of current input coefficients*, of the size $n \times n$. It corresponds to the matrix A well known from the static and dynamic Leontief models. Scrapping is included in the current input coefficients. Scrapping ratios are given exogenously.

$B(t)$ = the *matrix* of the technological input stock coefficients, i.e., of the coefficients denoting the technological engagement of assets, of the size $n \times n$. As a matter of fact, this corresponds to the matrix B of the dynamic Leontief model, but with an important difference. It comprises only the engagement of assets *technologically indispensable for production*. If, therefore, sector 8 is the producer of power generators and sector 2 is the producer of electric energy, $b_{8,2}$ denotes the quantity of generators directly participating in producing a unit of electric energy, *without the reserve* generators. Accordingly:

$$v_{ij}(t) = b_{ij}(t)\, r_j(t) + s_{ij}(t),$$

that is, the total input stock is composed of two parts, the *technological input stock* $b_{ij}(t)\, r_j(t)$ and the *slack input stock* $s_{ij}(t)$, the later serving as reserve.

Notation. We denote by J_t the following set: $J_t = \{(i, j);\, a_{ij}(t) = 0\}$. Of course, $b_{ij}(t) = 0$ if and only if $(i, j) \in J_t$. As usual, $\bar{J}_t$ stands for the complementum of J_t: $\bar{J}_t = \{(i, j);\, a_{ij}(t) > 0\}$.

We assume that J_t is time-invariant: $J_t = J$ for all t's. In addition, we assume that $v_{ij}(t) = 0$, $s_{ij}(t) = 0$, and $y_{ij}(t) = 0$ if $(i, j) \in J$. In this paper, for example, $V(t)$ is referred to as $\bar{J}$-positive, if $v_{ij}(t) > 0$; $(i, j) \in \bar{J}$, etc.

The pair of the two coefficient matrices, $\{A(t), B(t)\}$, is called the *real structure* of the system.

The Product Balance

The functioning of the real sphere of the model is easier to understand if, using scalar notation, the product balance of sector i in period t is surveyed:

$$r_i(t) = \sum_{j=1}^{n} a_{ij}(t)\, r_j(t) + w_i(t + 1) - w_i(t) + \sum_{j=1}^{n} [v_{ij}(t + 1) - v_{ij}(t)]. \quad (1.1)$$

Production is used for three purposes (for the sake of a simpler interpretation, let us assume that both production and stock are growing):

first term: current inputs into production;

second and third term: accumulation necessary for increasing the output stocks;

fourth term: the accumulation necessary to increase the input stocks.

The total of the output stocks and slack input stocks of the economy is called *buffer stock* or *slack*. Under ideal conditions (functioning completely without friction and disturbance) the economy could operate exclusively with the aid of the technological input stocks, without buffer stocks (slacks).

Assumptions About the Real Sphere

Let us sum up the assumptions about the real sphere.

R.1. *Closed Leontief-economy.* A product is turned out by a single sector and conversely, every sector releases but a single product. There is no substitution between inputs and no choice between alternative technologies. There are no joint products. There are no external resources nor final consumption. The consumption of the "labor sectors" of the economy appears as input, their work as output, and, therefore, they are not distinguished in the model from the other "common" sectors. The economy is irreducible.

R.2. *Homogenous linear input functions.* The current input into production and the technologically necessary engagement of assets are homogeneous linear functions of the volume of output.

R.3. *Exogenous scrapping ratios.* The scrapping ratio of the assets technologically engaged in production per period is exogenously given.

R.4. *Productivity of the system.* The system is capable of growth. Beyond current inputs there remains a surplus for the building up and expanding of stocks (output stocks, technological and slack input stocks). In other words, the spectral radius of the matrix $A(t)$ is uniformly smaller than unity for all t's: $\sigma\{A(t)\} < 1 - \kappa$ where κ is a positive number less than unity, which is independent of t.

R.5.N. *The real structure is time-invariant.* The coefficient matrices of the real structure are constant over time: $A(t) = A$, $B(t) = B$.

R.5.A. *The real structure is asymptotic.* The coefficient matrices of the real structure closely approach, after the lapse of a finite time, some *basic structure.*

More precisely, for each time-invariant real structure $\{A, B\}$ there exist an appropriately small number $\epsilon > 0$ and an appropriately large integer T, with the neighborhood of time-variant real structures $\{A(t), B(t)\}$

defined as follows: If $t > T$, then $\| A(t) - A \|$, $\| B(t) - B \| < \epsilon$, where $\| \cdot \|$ denotes some norm of a matrix (e.g., the sum of the absolute values of the elements).

A time-variant real structure is called asymptotic if it is contained in the neighborhood of at least one time-invariant real structure.

Our figure illustrate this ϵ-asymptotic property with the example of the coefficient $a_{ij}(t)$.

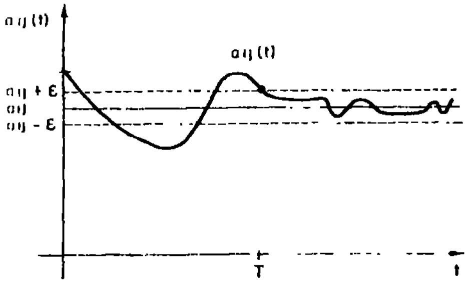

FIG. 1. Development of the input–output coefficients over time in an asymptotic economy.

Normative Control

Proceeding from the review of the real sphere to describing the control sphere, first of all the basic idea of *normative control* will be expounded.

It is assumed that for the decision making units of the economy (in our model the sectors) some *norms* are given which serve as guidelines in their behavior. In a dynamic system the norms appear perhaps in the form of *normative paths*. These are paths indicating the normative direction of the movement of the system.

The norms are developing as a result of historical and social processes and are fixed by conventions. In our paper the *origin* of the norms is not dealt with; this will be done in other studies. Here we simply assume they are a "given."

In this paper in the case of a Neumann economy the norms can be *computed* in a definite manner. But this is only a "trick" of the analysis of the system. It is not linked to any such institutional assumption as if in reality anyone "worked out" the norms of the system. In real life the norms are shaped tentatively by anonymous social processes.

Now if the norms (or the normative paths) are given, the decision makers may observe the differences between the actual and the normative values of the variables. Knowing this and counteracting the deviation they interfere and channel the system back towards the norms. Normative regulation takes place according to the principle of *negative feedback*, according to the following general *control formula*:

$$u(t) - u^*(t) = \varphi[t, x(t) - x^*(t)], \qquad \varphi(0) = 0, \tag{1.2}$$

$\varphi(t, \hat{x})$ monotonically decreases in $\hat{x}$; where $u(t)$ is the *control variable* of the system and $x(t)$ is the *state variable*. The asterisk distinguishes the normative value of the variable from its actual value without the asterisk. The form of the dependance φ must, of course, be specified in each concrete model.

In our model the production $r(t)$ and the purchase $Y(t)$ are the control variables, while the output stock $w(t)$ and the input stock $V(t)$ are the state variables.

Also a shortened notation will be introduced:

$$\begin{aligned} \hat{u}(t) &= u(t) - u^*(t), \\ \hat{x}(t) &= x(t) - x^*(t). \end{aligned} \tag{1.3}$$

These variables are called *deviation variables*.

Thus also the general control formula can be written in the form of

$$\hat{u}(t) = \varphi[t, \hat{x}(t)]. \tag{1.4}$$

Assumptions About the Control Sphere

Let us sum up the assumptions relating to the control sphere.

C.1. *Normative control.* The normative paths are feasible and given for the system: $r^*(t)$, $Y^*(t)$, $w^*(t)$, and $V^*(t)$.

Moreover, these variables as well as $S^*(t)$ are $\bar{J}$-bounded away from zero for all t's, and there exists a positive number κ which is smaller than every $\bar{J}$-coordinate of these variables for all t's.

C.2.N. *The normative path is a Neumann-path.*[4] A Neumann-path is denoted as follows:

$$\begin{aligned} \bar{r}(t) &= r_0\lambda_0{}^t, \\ \bar{Y}(t) &= Y_0\lambda_0{}^t, \\ \bar{w}(t) &= w_0\lambda_0{}^t, \\ \bar{V}(t) &= V_0\lambda_0{}^t, \end{aligned} \tag{1.5}$$

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[4] We do not consider its parallel, C.2.A, where the normative path is an asymptotic Neumann-path. See Theorem 1.

where λ_0 is the Neumann growth coefficient: $\lambda_0 > 1$. According to the assumption, the Neumann-path serves as a normative path, that is:

$$\begin{aligned} r^*(t) &= \bar{r}(t), \\ Y^*(t) &= \bar{Y}(t), \\ w^*(t) &= \bar{w}(t), \\ V^*(t) &= \bar{V}(t). \end{aligned} \tag{1.6}$$

Let us denote the matrix of the norms of the slack input stock per output by F and that of the norms of the output stock per production by $\langle g \rangle$:

$$\bar{S}(t) = F\langle \bar{r}(t) \rangle \quad \text{and} \quad \bar{w}(t) = \langle g \rangle \bar{r}(t), \tag{1.7}$$

and let us denote by H the matrix of the norms of buffer stocks: $H = F + \langle g \rangle$.

C.3. *Decentralization of the decisions.* With given norms decision is fully decentralized. Sector j decides on production r_j and for every i on the purchase y_{ij}.

There is a further degree of decentralization in the economy. In each sector production decision r_j and every purchase y_{ij} are made independently of each other. We may visualize the situation as if there were a production department responsible for production decisions and n purchase departments responsible for purchase decisions and these $n + 1$ departments were operating independently of each other.

C.4. *No memory.* Our decentralized control does not avail itself of a memory; it does not use as information the values of the variables taken in earlier periods.

C.5. *Decentralization of information.* With given norms information is completely decentralized. Sector j observes for making decisions only its own state variables. Let us call *autonomous* (*vegetative*) *control* the control scheme in which the assumptions C.3 and C.5 are asserted in combination.

Similarly to the decentralization of decision (see C.3) information is also decentralized within the sector among the production department and the purchase departments.

C.6. *Exclusivity of stock signal.* Assumption C.5 is realized in such a manner that the production department in sector j observes exclusively its own output stock w_j, and purchase department in sector j responsible for purchasing commodity i observes exclusively its own input stock v_{ij}.

C.7. *Linearity of the control rule.* Control is secured with the aid of *a linear formula. Its general form is* $\tilde{u}(t) = \alpha\hat{x}(t)$, *where* $\alpha = \langle\alpha_i\rangle$ *is the diagonal matrix of the speeds of adjustment.*[5] In formulas (2.3) and (2.4) below, $\langle d\rangle$ and E figure as matrices of speeds of adjustment.

C.8. *Homogeneity of adjustment speeds.* $\alpha_1 = \alpha_2 = \cdots = \alpha_m$.

Summary of the Model

Our dynamic system is described with the ensemble of the following four equations:

Output stock

$$w(t+1) = w(t) - Y(t)1 + r(t). \tag{2.1}$$

Slack input stock

$$V(t+1) = V(t) - A(t)\langle r(t)\rangle + Y(t). \tag{2.2}$$

Control of production

$$r(t) = r^*(t) - \langle d\rangle[w(t) - w^*(t)]. \tag{2.3}$$

Control of purchases

$$Y(t) = Y^*(t) - E \otimes [V(t) - V^*(t)]. \tag{2.4}$$

We have altogether $2n + 2n^2$ equations.

Equations (2.1)–(2.2) give the *laws of motion* and (2.3)–(2.4) give the *controls* of the system. The former deduce the new state variables of the period $(t+1)$ with the aid of the control and state variables of the period t. The latter deduce from the deviations between the actual and normative values of the state variables the deviations of the control variables from their normative values.

3. THEOREMS

Below we are going to review theorems, together with some short economic comments. The mathematical proofs of the theorems are to be found in the Appendix.

THEOREM 1[6] (The existence of a normative path). *Under Assumptions* R.1–R.4 *there exists a zone of feasible paths satisfying Assumption* C.1.

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[5] In Control Theory α is often called "control gain."

[6] We prove Theorem 1 without R.5.A, in its full generality.

THEOREM 1.N (The existence and the uniqueness of the normative Neumann-path). *Under Assumptions* R.1–R.4 *and* R.5.N *there exists a unique Neumann-path which satisfies Assumption* C.1 *and* C.2.N *and it is provided as a dominant solution of an eigenvalue–eigenvector problem.*

Remark. A word of explanation will be in order to clarify why C.1 is regarded as an *assumption* and Theorem 1 as a *conclusion*. In C.1 it is emphasized, that a feasible normative path is *given*, determined by historical and social processes, while in Theorem 1 it is emphasized that there exists a normative path which is *feasible* under the assumptions describing the technological conditions of the real sphere.

The relation between Assumptions C.1, C.2.N and Theorem 1.N is similar.

THEOREM 2.N (Buffer stock and growth rate). *Under Assumptions* R.1-R.4, R.5.N *and* C.2.N *let us compare economies* 1 *and* 2, *which are fully identical except for the norms of the buffer stock*: $h_{ij}^{(1)} \geqq h_{ij}^{(2)}$ *for every* (i, j), *and at least one component is definitely larger in economy* 1 *than in economy* 2. *In this case the growth coefficient of economy* 1 *is smaller than that of* 2: $\lambda_0^{(1)} < \lambda_0^{(2)}$.

Remark. With this theorem we state that in the world of our deterministic model the increase of the buffer stock norms slows down the growth of the economy. In real life the increasing of the buffer stock norms also has some advantages: it may help to fight the unexpected disturbances of production, make adaptation smoother, etc. But we cannot prove these advantages in this paper.

DEFINITION. The economy described in (2.1)–(2.4) is *stable*[7] if starting from an arbitrary initial point $[w(0), V(0)]$, all deviation variables, $\hat{w}(t)$, $\hat{V}(t)$, $\hat{r}(t)$, and $\hat{Y}(t)$, converge to zero as t converges to infinity.

The economy is *relatively stable* if starting from an arbitrary initial point $[w(0), V(0)]$, all relative deviation variables, $\hat{w}_j(t)/w_j^*(t)$, $\hat{v}_{ij}(t)/v_{ij}^*(t)$, $\hat{r}_j(t)/r_j^*(t)$, and $\hat{y}_{ij}(t)/y_{ij}^*(t)$, converge to zero as t converges to infinity.

Remarks. Since all normative variables are bounded away from zero, stability implies relative stability.

For the time being, up to the discussion of Theorem 5.A, we will neglect the question of positivity of variables $w(t)$, $V(t)$, $S(t)$, $r(t)$, and $Y(t)$. Since our economy is linear, local stability is equivalent to global stability.

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[7] In the theory of difference equations this property is often called *asymptotic stability*.

THEOREM 3.A[8] (Stability). *Conditions of stability of the system* (2.1)–(2.4) *under Assumptions* R.1–R.4, R.5.A, C.1, *and* C.3–C.7.

(i) Sufficient condition. *The system is stable if the decentralized feedback is* damped:

$$0 < d_j,\ e_{ij} \leq 1 \qquad 1 \leq i, j \leq n.$$

(ii) Necessary and sufficient condition. *Under the additional Assumption* C.8, *where* $\delta = d_j = e_{ij}$ $(1 \leq i, j \leq n)$, *the system is stable if and only if*

$$0 < \delta < 2/[1 + \sigma^{1/2}(A)].$$

THEOREM 3.N (Relative stability: Necessary and sufficient condition). *Under Assumptions* R.1–R.4, R.5.N, C.1, C.2.N, *and* C.3–C.8, *the system* (2.1)–(2.4) *is relatively stable if and only if*

$$-(\lambda_0 - 1)/(1 + \sigma^{1/2}(A)) < \delta < (\lambda_0 + 1)/(1 + \sigma^{1/2}(A)).$$

Remark. It is trivial that our system is stabilizable in a *centralized* way, under the above mentioned assumptions, without R.5.A, by the following rule:

$$\begin{aligned} \hat{r}(t) &= -[I - A(t)]^{-1}[\hat{w}(t) + \hat{V}(t)1], \\ \hat{Y}(t) &= -A(t)[I - A(t)]^{-1}[\hat{w}(t) + \hat{V}(t)1] - \hat{V}(t). \end{aligned}$$

In fact, all deviation variables become zero if $t \geq 1$.

The major innovation in our analysis is the statement that the stabilization of a strongly coupled system like (2.1)–(2.4) can be assured in a *decentralized* way as well. We used Assumption R.5.A for that purpose. It is still unanswered if our system is stabilizable in a decentralized way, without R.5.A.

DEFINITION. We call systems with identical real structures $\{A, B\}$ *similar* economies.

THEOREM 4 (Adjustment speeds and convergence rates.[9]) *We compare similar economies, differing only in adjustment speeds, under Assumptions* R.1–R.4, R.5.N, C.1 *and* C.3–C.7. *We consider the following cases*:

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[8] Of course, Theorem 3.A is valid for a Neumann economy as well, since a Neumann-economy is *a fortiori* an asymptotic economy. The same remark applies to Theorem 5.A.

[9] The term "convergence rate" is well known in the literature of difference equations. For special definition in our case see the proof of Theorem 3.A.

(i) *We compare two similar economies*: 1 *and* 2. *The adjustment speeds are damped*: $0 < d_j^{(1)} \leq d_j^{(2)} \leq 1$ *and* $0 < e_{ij}^{(1)} \leq e_{ij}^{(2)} \leq 1$ *for every j and* $(i, j) \in \bar{J}$, *and at least one adjustment speed is smaller in economy* 1 *than in economy* 2. *In this case the convergence rate of the economy* 1 *is smaller than that of* 2: $1/\hat{\lambda}^{(1)} < 1/\hat{\lambda}^{(2)}$.

(ii) *We consider all similar economies with damped adjustment speeds. The convergence rate is maximal if and only if*

$$d_j = 1 = e_{ij} \quad \textit{for} \quad j = 1, 2, \ldots, n \quad \textit{and} \quad (i, j) \in \bar{J}.$$

(iii) *We consider all similar economies with homogenous adjustment speeds* (C.8). (*In that case we do not assume, as in* (ii), *that the adjustment speeds are damped.*) *The convergence rate is maximal if and only if*

$$\delta = 1.$$

Remarks. Alternative assumptions (ii) (*damped* adjustment speeds) and (iii) (*homogenous* adjustment speeds) lead to the same result: we achieve fastest convergence by adjustment speeds all equal to unity. The maximal convergence rate is equal to $1/\sigma^{1/2}(A)$.

Our theorem is independent of the choice of the normative path, and, *a fortiori*, of F and $\langle g \rangle$.

We apply neither suffix N, nor suffix A denoting Theorem 4, for the following reasons: We assume a time-invariant real structure, i.e., the system is a Neumann-economy in this respect. We do not apply, however, Assumption C.2.N, i.e., we do not assume that the normative path is a Neumann-path. In this respect the Theorem is more general.

DEFINITION. The system is *viable* if none of its variables is negative, and in each period at least one production variable is positive. We call the *viable initial neighborhood* the set of all initial points which provide viable systems.

THEOREM 5.A (Viability). *Under the Assumptions* R.1–R.4, R.5.A, C.1, *and* C.3–C.7, *in every* (*relatively*) *stable economy we can find a viable initial neighborhood.*

Remarks. The theorem holds for both the Neumann-economy and the asymptotic economy. For the Neumann-economy, relying on the additional Assumption C.2.N, we have a simple constructive method for determining a viable initial neighborhood. (See the proof of the Theorem.)

The common practical interpretation of Theorems 3.A and 5.A is the following: Under definite conditions the control described in this paper is viable and is suited for securing (relative) stability.

4. ABOUT THE ECONOMIC INTERPRETATION OF THE THEOREMS

Elementary Quantitative Adaptation

Like its predecessors, the Kornai–Martos model and the models of Dancs Hunyadi–Sivák and of Virág, the model described in this paper does not claim to be a *general* model of the regulation of the economy.

The regulation of economic systems is a complex problem, a task performed in combination by various control mechanisms and submechanisms. A part of these control mechanisms control primitive, short-term quantitative adaptation processes, with very simply decentralized decision and information structures. The Kornai Martos article called this type of control "autonomous" or "vegetative" control. Our paper intends to contribute to the analysis of exclusively such "elementary," "primitive," "autonomous," "vegetative" control mechanisms.

At the same time, in every economy there are also mechanisms operating, which fulfil more complicated control tasks. Such are, for example, the qualitative development of production and consumption, the introduction of new products, the combination of various resources, and, in this context, the choice between alternative technologies; decision on the rate of accumulation, long-term adaptation to lasting changes, and so on. Obviously, the mechanism described in the present study is not suited for fulfilling the more complicated control functions.

The World of Leontief-Neumann Dynamic Models and Economic Reality

It is well known, that the basic assumptions of dynamic Leontief models, resp. Neumann-models are very strong; they are gross simplifications of economic reality. The authors do not deny that they have been forced to accept these assumptions in the present stage of research while grinding their teeth. For the sake of some self-assuredness, it is worth while to follow the reasoning as below:

1. Some of our theorems require more restrictive, and some others less restrictive sets of assumptions. Let us consider the most restrictive one at first: time invariant real structure *and* a Neumann-path, serving as the normative path of control. This would be an almost unendurably "bad" model of the economic system if we were going to use it for *planning* the real sphere. By force of the assumptions themselves, we eliminate the substance of the planning problem: the choice of the relative ratios of the sectors and the technologies. The model is much less "bad" for our purposes (for the theoretical analysis of the control mechanism) and precisely of the control mechanism regulating elementary primitive quantitative adaptation.

Not only in our model, but also in reality, the decision makers are inclined to apply in routine-like quantitative adaptation simple recipes, rules of thumb, which do not take into account the structural shifts in the farther economic environment, but react merely to the directly observable signals.

The elementary quantitative adaptation of the Neumann-economy described in the present study operates with the aid of such stock normatives which resemble the form of stock normatives observable in a real economy. The norms are fixed in the form of a quotient of output stock per output or input stock per purchase ("To the production or purchase of how many months does the stock correspond"). We have succeeded in working out such simple forms of norms only for the Neumann-economy up to now.

True, this elementary control mechanism might lead to incorrect decisions if considerable qualitative changes occur, shifts in proportions in the composition of final resources and final consumption, as well as in technology. In such cases the *limits* of the elementary control described in our paper will have been reached. On the one hand other, higher regulators must enter, but on the other hand perhaps the norms must be changed. We feel, however, that this is not a deficiency of the model. We intended to present not a *universal* mechanism equally suited for fulfilling any kind of control task, but a special one capable of fulfilling a few elementary control tasks, and we think it is natural that the control parameters figuring in it, the norms and the speeds of adjustment reaction, must be subjected from time to time to revision.

2. The assumptions about the asymptotic economy are less strong than those treated in paragraph 1. Also, the asymptotic economy is an abstraction, but one standing nearer to reality. The technological changes, shifts in proportions which, as regards their lasting trend, show some definite monotonous regularity, can be described with good approximation as an approach to some basic structure. If, for example, with primary energies there is a monotonous tendency for a growing share of nuclear energy, this can be described as approaching some ratio for nuclear to conventional primary energies.[10]

3. We applied even less restricted sets of assumptions to our two most relevant propositions, to Theorem 3.A on stability, and Theorem 5.A

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[10] Regarding a longer historical development, the share of coal first increased when wood was pushed to the background, and then it diminished under the effect of the advance of oil and natural gas, and now, at least transitorily, it will be growing again in the wake of the higher oil and gas prices. Such a nonmonotonous shift in proportions cannot be adeqately described with the aid of the asymptotic assumption.

on viability. Here we assume time-variant asymptotic real structure, but we do not need any restriction on the normative paln (apart from feasibility). *The system can be controlled in a decentralized way, and it will be stable and viable even when the normative path is not a Neumann-path.*

That is the reason we could not apply to a larger extent the literature on Neumann-growth and the turnpike theorems. Namely, we posed a different set of *questions* to be answered. Turnpike theorems belong to *growth* theory; our theorems belong to the theory of *control of economic systems*; both being markedly different branches of economics. Turnpike theorists are searching for the solution of a *normative* problem: optimal conditions of growth. Our study, on the contrary, is more a *descriptive-explanatory* investigation. We are exploring elementary control mechanisms which are able to regulate any kind of economies, including those which are *nonoptimal*, i.e., do not move along (or close to) a Neumann-path. To find an answer to this broad question, we cannot restrict our study to the much narrower case of optimal growth paths.

5. COMPARISON WITH OTHER MODELS

As has been mentioned in the Introduction, this paper has been born from cross-breeding three schools: modeling the control mechanism of the economy, the literature of the Leontief-Neumann models, and the economic application of mathematical control theory. It would lead too far if comparisons were made even with the main models of all three schools. Therefore, the circle will be essentially narrowed down. Exclusively those models will be dealt with which examine the *control mechanism of dynamic economic systems*. (Surprisingly few models can be listed in this category.)

The comparison is made in the form of a table. The columns show the different models, the rows the criteria of the models.

Although the table speaks for itself, some additional remarks are in order.

In columns 1–4 and 7 of the table there are models which are closely related: they are all linked to the Kornai–Martos study. Therefore, they are easily comparable. The studies of Lovell [22] and McFadden [23] set out from rather different questions. Therefore, these have been first adapted to the problem examined by us and only then compared to the others.

Only a single attempt has been made so far at a stochastic analysis of the problem: in the article of Virág. One of the most important tasks is to extend research in this direction.

The article by Kornai and Martos [16, 17] and in its wake Virág [33] and Dancs, Hunyadi, and Sivák [8]) did not insist on a real structure

TABLE I

	Kornai and Martos [16, 17]	Virág [33]	Dancs, Hunyady, and Sivák [8]	Bródy [5]	Lovell [22]	McFadden [23]	Kornai and Simonovits
0 Deterministic 1 Stochastic	0	1	0	0	0	0	0
0 Continuous over time 1 Discrete ove time	0	0	0 and 1	0 and 1	1	1	1
0 Engagement of assets is not represented 1 Engagement of assets is represented	0	0	0	0	1	0	1
0 Structure is constant over time 1 Structure is variable over time	1	1	1	0	0	0	0 and 1
0 Stagnating 1 Growing	1	1	1	0	0	1	1
0 Centralized 1 Decentralized	1	1	0	1	0 and 1	1	1
0 Memory 1 No memory	0	0	0	0 and 1	0	1	1

constant time, but allowed its arbitrary changes in time. The present study applies, on the one hand, a stronger assumption when it sets out from a real structure either constant in time or asymptotic, but on the other hand, it has a compensating merit: the system of norms is very simple in the case of the Neumann-economy; it takes a form similar to the stock norms in real life (stocks as a fixed proportion of production or purchase).

The article by Kornai and Martos handled time as a continuous variable and did not describe any lags. The model of Dancs, Hunyadi, and Sivák and the present paper handle time as a discrete variable, with the advantage that it can be directly simulated by a computer. One of the essential differences between the model of Dancs, Hunyadi, and Sivák and the present one is that in the former the control variables can be determined only in possession of centralized information, while in the latter the system is decentralized also from the point of view of information. The former is not autonomous (vegetative) regulation, the latter is one.

From among the studies listed, the present one is the first which describes pure stock signals. In the Kornai–Martos study and in the related other works, in addition to stocks sales also figured among the signals.

There is no way for a simple confrontation of the "centralized" control schemes with the "decentralized" ones, since this depends, among other things, on how the norms come about. (Are they not determined perhaps by some centralized process?) Therefore, in the comparison the sign 1 is put where, in possession of given norms, both the decision and the information serving as a basis of the decision are strictly decentralized.

In conclusion, we should like to stress that the table also indicates the further direction of research. We have made efforts to try the cases denoted by both 1 and 0, from the viewpoints of every criterion. In addition, we should also like to expand research in directions not shown in table, to include scarce primary resources and alternative technologies, for example.

Two further comments on the comparisons.

There are some important theorems related to *decentralized* control, (e.g., in Aoki [1].) These theorems, however, cannot be applied directly to the discussion of Leontief-Neumann dynamic systems, like the economies described in our paper. Therefore we did not include that part of the literature in our comparative table.

In addition, there is a huge literature on *centralized* control, among others on the so-called tracking problem. Unfortunately, *decentralization* of control raises new difficulties in the mathematical nature of the issues, which exclude direct application of the tracking theorems.

APPENDIX: MATHEMATICAL PROOFS OF THE THEOREMS

Proof of Theorem 1. We shall consider the product balance (1.1) for a normative path:

$$r^*(t) = A(t)\,r^*(t) + w^*(t+1) - w^*(t) + V^*(t+1)1 - V^*(t)1. \tag{6.1}$$

If $w^*(t)$ and $V^*(t)$ are given (and positive), then every $w^*(t+1)$ and $V^*(t+1)$ is *reachable* by a unique positive *decision-pair* $r^*(t)$ and $Y^*(t)$ if

$$w^*(t+1) \geq w^*(t) \quad \text{and} \quad V^*(t+1) \geq V^*(t), \tag{6.2}$$

and

$$B(t)\langle[I - A(t)]^{-1}\cdot[w^*(t+1) - w^*(t) + V^*(t+1)1 - V^*(t)1]\rangle \leqq V^*(t). \tag{6.3}$$

To prove this let us mention that (6.2) assures the positivity of

$$r^*(t) = [I - A(t)]^{-1}[w^*(t+1) - w^*(t) + V^*(t+1)1 - V^*(t)1],$$

the formula provided by (6.1) and R.4. Furthermore, $Y^*(t)$ is uniquely determined by (2.2) and by $r^*(t)$, $V^*(t)$, and $V^*(t+1)$. Its $\bar{J}$-variables are positive, its J-variables are zero.

Finally, $V^*(t) \geqq B(t)\langle r^*(t)\rangle$ is equivalent to (6.3). Since $[I - A(t)]^{-1}$ is positive, $B(t)$ and $V^*(t)$ are $\bar{J}$ positive, sufficiently small increases in all normative stocks are feasible. Q.E.D.

Proof of Theorem 1.N. If the N-path exists, by substituting into the product balance the relations of C.2.N and dividing it by λ_0', the following equation is obtained:

$$r_0 = Ar_0 + (\lambda_0 - 1)\langle g\rangle r_0 + (\lambda_0 - 1)(B + F)\,r_0. \tag{6.1.N}$$

After rearrangement this leads to the eigenvalue-eigenvector problem

$$(1/(\lambda_0 - 1))\,r_0 = (I - A)^{-1}(B + H)\,r_0.$$

It is well known (see, e.g., Bródy [4, Appendix 1]) that there exists a unique (r_0, λ_0) which is positive and r_0 satisfies the norm $1'r_0 = 1$. r_0 uniquely determines w_0, v_0, and S_0 according to C.2.N and then also Y_0 is uniquely determined, because of (2.2). Each of v_{ij}^0, s_{ij}^0, and y_{ij}^0 is zero if and only if $(i,j) \in J$.

Proof of Theorem 2.N. From the theory of positive matrices (cf. Morishima [26] and Bródy [4]) the following lemma is known:

LEMMA I. *If U_1, U_2 are irreducible matrices with nonnegative elements of the size $n \times n$, for which $U_1 \geq U_2$, then $\sigma(U_1) > \sigma(U_2)$.*

In our case $U_i = (I - A)^{-1}(B + H^{(i)})$ and $\sigma(U_i) = 1/(\lambda_0^{(i)} - 1)$ $(i = 1, 2)$. $H^{(1)} \geq H^{(2)}$ implies $U_1 \geq U_2$, hence by Lemma I $\sigma(U_1) > \sigma(U_2)$. Since $\lambda_0^{(i)} = 1 + 1/\sigma(U_i)$, $\lambda_0^{(1)} < \lambda_0^{(2)}$. Q.E.D.

The following *monotonicity* property plays a great role in examining both (relative) stability and viability.

LEMMA II. *If $0 < d_i, e_{ij} < 1$ $(1 \leq i, j \leq n)$, then*

$$w^{(1)}(0) \geq w^{(2)}(0) \quad \textit{and} \quad V^{(1)}(0) \geq V^{(2)}(0)$$

imply the inequalities $w^{(1)}(t) \geq w^{(2)}(t)$ and $V^{(1)}(t) \geq V^{(2)}(t)$ for $t = 1, 2, \ldots$ where the structures and the speeds of adjustment of the systems (1) *and* (2) *are identical, only their initial states differ to the advantage of* (1).

Proof of Lemma II. Let us substitute the normative path into (2.1)–(2.2) and deduct it from the original:

$$\hat{w}_j(t+1) = \hat{w}_j(t) - \sum_{k=1}^{n} \hat{y}_{jk}(t) + \hat{r}_j(t),$$

$$\hat{v}_{ij}(t+1) = \hat{v}_{ij}(t) - a_{ij}(t)\,\hat{r}_j(t) + \hat{y}_{ij}(t).$$

Let us substitute (2.3)–(2.4) into our new equations:

$$\hat{w}_j(t+1) = (1 - d_j)\,\hat{w}_j(t) + \sum_{k=1}^{n} e_{jk}\hat{v}_{jk}(t), \tag{6.4}$$

$$\hat{v}_{ij}(t+1) = d_j a_{ij}(t)\,\hat{w}_j(t) + (1 - e_{ij})\,\hat{v}_{ij}(t). \tag{6.5}$$

From the conditions of Lemma II it follows immediately that the coefficient matrix of the iterative formulas (6.4)–(6.5) of the dimension $(n^2 + n - |J|)$[11] is nonnegative, regular, from which the monotonicity of the "system of deviations" is immediately to be seen. Since the "system of deviations" is a system reduced by means of the normative path, the original system is also monotonous.

As a contradiction, we shall assume that our matrix in (6.4)–(6.5) is reducible, at least for one t. Then, by definition, we can divide our J-variables into two classes: the first class is referred to as the class of *free* variables, while the second class is referred to as the class of *zero* variables; and at this classification each transformed zero variable remains a zero variable, and the second class is nonempty.

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[11] $|J|$ denotes the number of the elements of J.

Let $K = \{j: \hat{w}_j(t) \text{ is free}\}$ and $\bar{K} = \{j: \hat{w}_j(t+1) = 0 = \hat{w}_j(t)\}$. By (6.4), if $j \in \bar{K}$ and $(j, k) \in \bar{J}$, then $\hat{v}_{jk}(t) = 0$. By (6.5), if $j \in K$ and $(i, j) \in \bar{J}$, then $\hat{v}_{ij}(t)$ is free.

K is nonempty, too, because our matrix is nonzero. Since matrix $A(t)$ is irreducible, there exists at least one $(i, j) \in \bar{J}$. for which $i \in \bar{K}$ and $j \in K$. Thus $\hat{v}_{ij}(t)$ is zero as well as free, which is a contradiction. Q.E.D.

Proof of Theorem 3.A. First we shall prove Theorem 3.A for time-invariant systems.

(i) Let $(\hat{\lambda}, \hat{w}, \hat{V})$ be the normed positive dominant solution of the linear system (6.4)-(6.5). We shall call $1/|\hat{\lambda}|$ the (asymptotic) *convergence rate*. Here the matrix of coefficients is nonnegative and irreducible, that is, according to Frobenius' theorem it has a unique positive dominant solution.

With the aid of the positive dominant solution we can give a simple constructive condition for the (relative) stability and the viability of the system.

To prove stability, we need to prove $\hat{\lambda} < 1$. Since $(\hat{\lambda}, \hat{w}, \hat{V})$, a dominant solution of (6.4)–(6.5), is the unique positive solution, it is sufficient to prove the existence of a positive solution with $\hat{\lambda} < 1$.

Substitute $(\hat{\lambda}, \hat{w}, \hat{V})$ into (6.4)-(6.5). After a straightforward calculation we get the following n-dimensional, nonlinear fix-point problem:

$$\hat{w}_j = \sum_{k=1}^{n} \frac{d_j e_{jk}}{(\hat{\lambda} - 1 + d_j)(\hat{\lambda} - 1 + e_{jk})} a_{jk} \hat{w}_k , \qquad (6.6)$$

$$\hat{w}_j > 0, \qquad j = 1,\dots, n \qquad \text{and} \qquad \hat{\lambda} > 0.$$

Let us consider the following characteristic value–vector problem:

$$\sigma_\lambda w_j = \sum_{k=1}^{n} \frac{d_j e_{jk}}{(\lambda - 1 + d_j)(\lambda - 1 + e_{jk})} a_{jk} w_k , \qquad (6.7)$$

$$w_j > 0, \qquad j = 1, 2,\dots, n \qquad \text{and} \qquad \lambda > 0.$$

We underline that for $\lambda = \hat{\lambda}$ the second problem is reduced to the first one. Let $\mu = \max\{1 - d_j , 1 - e_{jk} ; 1 \leqq j \leqslant n; (j, k) \in \bar{J}\}$.

By $0 < d_j , e_{jk} \leqq 1$, $0 \leqslant \mu < 1$. If $\lambda > \mu$; then each coefficient of the problem is nonnegative and their matrix is irreducible, with spectral radius σ_λ.

At $\lambda = \mu + 0$ at least one coefficient is equal to $+\infty$, hence $\sigma_{\mu+0} = +\infty$, too. At $\lambda = 1$, $\sigma_1 = \sigma(A)$, which is smaller than 1. Since σ_λ is continuous, by the intermediate-value theorem (Bolzano theorem) there exists a $(\hat{\lambda}, \hat{w}, \hat{V})$ with positive elements and $\hat{\lambda} < 1$. Q.E.D.

(ii) To get a necessary and sufficient condition on stability, we shall restrict our analysis to *homogeneous adjustment speeds.* Let $d_j = \delta \cdot e_{jk}$, $1 \leq j, k \leq n$, then we get from our fixpoint problem (6.6) the following characteristic value–vector problem:

$$((\lambda - 1 + \delta)^2/\delta^2)w = Aw.$$

Since we have dropped the assumption of *damped adjustment speeds,* we do not know a priori if the unique positive solution is a dominant one. Let $\{\nu_i\}_{i=1}^n$ be the characteristic values of the matrix A and let $(\{\lambda_i^{(h)}\}_{i=1,h=1}^{n\;\;2})$ be the corresponding characteristic values of the system (6.4)–(6.5). Obviously

$$\lambda_i^{(1,2)} = 1 - \delta(1 \pm \nu_i^{1/2}) \qquad 1 \leq i \leq n. \tag{6.8}$$

Now $|\hat{\lambda}| = \max\{|\lambda_i^{(h)}| ; 1 \leq i \leq n, h = 1, 2\}$. Evidently, if $\delta \leq 0$, then all $|\lambda_i^{(1)}|$'s are greater than or equal to 1, i.e., the system is *unstable.*

$$\text{If } \delta > 1, \text{ then } |\hat{\lambda}(\delta)| = |1 - \delta[1 + \sigma^{1/2}(A)]| \tag{6.9}$$

and this is smaller than unity if and only if $\delta < 2/[1 + \sigma^{1/2}(A)]$. Q.E.D.

Proof of Theorem 3.A (for time-variant systems). It is well known that the stability theorems referring to the linear difference and differential equations can be easily extended from constant matrices to asymptotic ones, as proved by Poincare (cf. Gelfond [12]) and Perron (cf. Bellman [3, Chap. II, Theorem 2]). Relying on them, we state without further proof

LEMMA III. *Let the $z(t+1) = Uz(t)$ first-order linear system of difference equations with constant coefficients be stable, that is $\sigma(U) < 1$. Then there exists an adequately small $\epsilon > 0$ and an adequately large integer T, for which with an arbitrary series of matrices $\{U(t)\}_{t=0}^{\infty}$, the system of equations $\tilde{z}(t) = U(t)\,\tilde{z}(t)$ is equally stable, provided that $\| U(t) - U \| < \epsilon$ for every $t > T$.*

Proof of Theorem 3.N. We return to the Neumann-economy with homogenous adjustment speeds. Since $0 < \delta \leq 1$ assures stability as well as relative stability, now we turn to $\delta \leq 0$ and $\delta > 1$. From (6.8) it follows that in both cases $\hat{\lambda}(\delta) = -1 + [1 + \sigma(A)^{1/2}]\delta$. Relative stability is equivalent to $|\hat{\lambda}| < \lambda_0$, thus $\hat{\lambda}(\delta_{\text{inf}}) = -\lambda_0$ and $\hat{\lambda}(\delta_{\text{sup}}) = \lambda_0$ define the infimum and the supremum of those homogenous adjustment speeds which provide relative stability. The last three equations provide the results. Q.E.D.

Proof of Theorem 4. (i) The proof is rather similar to that of Theorem 2.N, therefore we emphasize only the new aspects.

Considering again (6.7), we see that each coefficient of the matrix is a decreasing function of d_j as well as of e_{jk}, $(j, k) \in \bar{J}$. Hence comparing our two economies we get $\sigma_\lambda^{(1)} < \sigma_\lambda^{(2)}$ for every $\lambda > \mu^{(1)}$. Hence $\hat{\lambda}^{(1)} > \hat{\lambda}^{(2)}$.

(ii) Consequently, the optimal damped adjustment speeds are all maximal, i.e., they are equal to unity.

(iii) Now we drop the assumption of damped adjustment speeds but assume homogenous adjustment speeds. If $\delta < 0$, then the system is unstable. If $0 < \delta \leq 1$, then we have seen just before that $\delta = 1$ is the optimum. If $\delta > 1$ then (6.9) implies that $|\hat{\lambda}(\delta)| > |\hat{\lambda}(1)|$. Q.E.D.

Remark. It would have been rather cumbersome to apply McFadden's theorem, and anyway the convergence rate would have been much less than our optimum, because the adjustment speeds would have been almost zero.

Proof of Theorem 5.A. It is generally true that, in the case of a relatively stable system of difference equations, if the equilibrium path is at every point of time bounded from below by a positive vector which is constant over time, there exists an initial neighborhood from which we can always set out on a path which is positive all the way along. Q.E.D.

Proof of Theorem 5.N. For the case of the Neumann-economy a constructive method will be given to determine an initial neighborhood that is maximal, in a sense.

We need the following quantities:

$$\omega = \min_{(i,j)\in J} \frac{s_{ij}^0}{b_{ij} d_j \hat{w}_j}\,\eta \quad (0 < \eta < 1), \qquad \tilde{\alpha} = \min_{1 \leq j \leq n} \frac{w_j^0}{\hat{w}_j},$$

$$\alpha = \min\{\omega, \tilde{\alpha}\}, \qquad \beta = \min_{1 \leq j \leq n} \frac{r_j^0}{d_j \hat{w}_j},$$

$$\gamma = \min_{(i,j)\in J} \frac{s_{ij}^0 - \alpha b_{ij} d_j \hat{w}_j}{\hat{v}_{ij}}, \qquad \xi = \min_{(i,j)\in J} \frac{y_{ij}^0}{e_{ij} \hat{v}_{ij}},$$

$$w^{(m)} = w_0 - \alpha \hat{w}, \qquad w^{(M)} = w_0 + \beta \hat{w},$$

$$V^{(m)} = V_0 - \gamma \hat{V} \quad \text{and} \quad V^{(M)} = V_0 + \xi \hat{V}.$$

LEMMA IV. *The initial neighborhood defined with the aid of the inequalities* $w^{(m)} \leqq w(0) \leqq w^{(M)}$ *and* $V^{(m)} \leqq V(0) \leqq V^{(M)}$ *is not empty but viable. The increase of any of the* (α), β, γ, *and* ξ *in* ($w^{(m)}$), $w^{(M)}$, $V^{(m)}$, *and* $V^{(M)}$ *will cause the ceasing of viability right at the start, (if* $\tilde{\alpha} < \omega$*).*

Proof of Lemma IV. *Let us pass to the "system of deviations." Then our initial neighborhood is* $-\alpha\hat{w} \leq \hat{w}(0) \leq \beta\hat{w}$ and $-\gamma\hat{V} \leq \hat{V}(0) \leq \xi\hat{V}$. *Because of the monotonicity.* $-\alpha\hat{w}\hat{\lambda}^t \leq \hat{w}(t) \leq \beta\hat{w}\hat{\lambda}^t$ and $-\gamma\hat{V}\hat{\lambda}^t \leq \hat{V}(t) \leq \xi\hat{V}\hat{\lambda}^t$ *for every* t. *Owing to the relative stability of the system* $0 < \hat{\lambda} < \lambda_0$ and, because of the definition,

$$-w_0 \leq -\alpha\hat{w}, \qquad \beta\hat{w} \leq \langle d_j^{-1}\rangle r_0,$$
$$-\alpha B\langle d_j\hat{w}_j\rangle - S_0 \leq -\gamma\hat{V}, \qquad \xi\hat{V} \leq [e_{ij}^{-1}] \otimes Y_0 .$$

It follows that

$$-w_0\lambda_0^{\,t} \leq \hat{w}(t) \leq \langle d_j^{-1}\rangle r_0\lambda_0^{\,t}$$

and

$$-B\langle d_j\hat{w}_j(t)\rangle - S_0\lambda_0^{\,t} \leq \hat{V}(t) < [e_{ij}^{-1}] \otimes Y_0\lambda_0^{\,t} \qquad (t > 0),$$

which, according to (2.3) and (2.4) secures, beside the positivity of $w(t)$ and $S(t)$, also the positivity of $r(t)$, $V(t)$, and $Y(t)$ for $t > 0$. Q.E.D.

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The General Equilibrium Theory of Tax Incidence under Uncertainty

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INTRODUCTION

Meade [8], Wells [14], Johnson [6], Musgrave [10], Harberger [5], and Mieszkowski [9] have presented several general equilibrium results on tax incidence under certainty. The purpose of this paper is to analyze the implications of uncertainty in production for these well-known results on tax incidence under general equilibrium. We employ a two-sector, two-factor model with uncertainty in the production function of one sector and assume that producers are expected utility maximizers. The results now turn on the firms' attitude toward risk, as summarized below.

Partial Factor and Commodity Taxes

The effect of partial taxes on factor prices is less if producers are risk averse than if they are risk neutral or if they produce under conditions of certainty.

1. *Certain sector.* The certainty results, that (a) the combined effect of equal-rate partial factor taxes on factor prices is the same as an equal-rate commodity tax, and (b) equal-yield partial taxes have the same effect on factor prices, continue to hold.

2. *Uncertain sector.* The above certainty results are altered as follows if producers are risk averse: (a) the more (less) intensive factor gains (loses) from the replacement of partial factor taxes by an equal-rate commodity tax; (b) (i) capital (labor) gains (loses) from the replacement of a partial capital (labor) tax by an equal-yield partial labor (capital) tax, and (ii) the

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replacement of a partial factor tax by an equal-yield partial commodity tax results in the intensive factor gaining if it is the taxed factor initially.

General Factor, Income, and Sales Taxes

(1) In the case of certainty (a), under a factor tax the taxed factor bears 100% of the burden of the tax and (b) the burden of an income tax is borne by each factor in proportion to its initial share in income. These results continue to hold under uncertainty, provided producers exhibit constant absolute risk aversion. Given declining (increasing) absolute risk aversion, the relatively intensive factor is worse (better) off under these taxes than when risk aversion is constant; (c) the differential incidence on factor prices of general factor taxes is the same under uncertainty as under certainty, regardless of the attitude towards risk.

(2) (a) Under certainty, a general sales tax is borne by the factors in proportion to their initial shares in income; but under risk aversion, the more (less) intensive factor in the uncertain sector bears a less (more) than proportionate burden. (b) Under certainty, the differential incidence of a sales tax and an income tax is zero; however, under uncertainty, given decreasing absolute risk aversion, the replacement of an income tax by an equal-yield sales tax makes the more (less) intensive factor in the uncertain sector better (worse) off.

In our model we assume a small country because the assumption of a large country required the specification of a demand equation for the determination of product prices. Due to the presence of uncertainty in the production of one good, its supply is random, and because demand has to equal supply in order to clear markets, product price will be a random variable; the ultimate value of which will be known only after all production decisions are made. However, producers make their production decisions on the basis of given product prices. One way out of this impasse is to assume nonperfect competition on the part of producers, and that they know consumer demand functions, which allows them to treat product prices as random variables. This makes the problem intractable. The other solution, which we have chosen, is to assume a small country with given product prices.

In what follows, we present our model in Section 1, the results on incidence in Section 2, and our conclusions in Section 3.

1. THE MODEL

We make the usual neoclassical assumptions regarding the conditions of production, i.e., we have perfect markets, full employment and inelastic supplies of factors of production, linear, homogeneous, and quasi-concave production functions, irreversible factor intensities, and perfect intersectoral mobility of factors.

We assume that the two-sector economy produces X_1 in the certain and X_2 in the uncertain sector, respectively. The former uses labor L_1 and capital K_1, the latter, labor L_2 and capital K_2, where L and K are the total endowments of labor and capital, respectively. The production function in the uncertain sector is of the form

$$X_2 = \alpha F_2(K_2, L_2) = \alpha L_2 f_2(k_2), \tag{1}$$

where $k_j = K_j/L_j$, and α is a random variable with density function $g(\alpha)$ and an expected value μ. This simple form of uncertainty in the production function has been used by Bardhan and Srinivasan [1], Smith and Rothenberg [12], Sandmo [13], and Batra [3] on the grounds that a more complicated form would not yield determinate results. The random variable α may incorporate factors such as weather, accidental breakdowns, etc., the decisions regarding factor demands having to be made prior to the resolution of α. We assume that all producers in the second sector X_2 have identical utility functions and that they seek to maximize expected utility from profits. The assumption that firms maximize expected utility allows us to describe in a simple way risk-averse, risk-neutral, or risk-preferring behavior, and also to appeal to certain desirable properties, such as nonincreasing absolute risk aversion when signing results. That is, there are no limitations on the utility function, as there would be if we assumed that firms maximized expected profits. We further justify use of the expected utility hypothesis when analyzing purely competitive firms by pointing out its previous use by Penner [11], Baron [2], Sandmo [13], Batra and Ullah [4], etc., and citing Mao's [7] empirical investigation with business executives that implied firms are not risk neutral.

If we let p_j, r_j, and w_j $(j = 1, 2)$ be the price, rental rate, and wage rate in the jth sector, respectively, profit in X_2,

$$\Pi_2 = p_2X_2 - r_2K_2 - w_2L_2 = p_2\alpha F_2(K_2, L_2) - r_2K_2 - w_2L_2. \tag{2}$$

Expected utility from profits is given by

$$E[U_2] = E[U_2(\Pi_2)],$$

where $U_2(\Pi_2)$ is the Von Neumann–Morgenstern utility function and E is the expectation operator. Since producers take prices as given and seek to maximize expected utility from profits by choosing L_2 and K_2, the first-order conditions for a maximum are given by

$$\partial E(U_2)/\partial L_2 = E[U_2'(\Pi_2)(p_2\alpha F_{L_2} - w_2)] = 0, \tag{3}$$

and

$$\partial E(U_2)/\partial K_2 = E[U_2'(\Pi_2)(p_2 \alpha F_{K_2} - r_2)] = 0, \tag{4}$$

where αF_{K_2} and αF_{L_2} are the marginal products of capital and labor, respectively.

In the first sector, in the absence of uncertainty, the production function and emerging first-order conditions are

$$X_1 = F_1(K_1, L_1) = L_1 f_1(k_1), \tag{5}$$

$$w_1 = p_1 F_{L_1} \tag{6}$$

and

$$r_1 = p_1 F_{K_1}. \tag{7}$$

From (3) and (4) we have K_2 and L_2 as functions of p_2, r_2, and w_2. Similarly, from (6) and (7) we obtain K_1 and L_1 as functions of p_1, r_1, and w_1. In the absence of taxation and given perfect markets we have $w_1 = w_2$, $r_1 = r_2$, $p_1 = p^1$, and $p_2 = p^2$, where p^1 and p^2 are the given world prices.

Under full employment,

$$L_1 + L_2 = L \tag{8}$$

and

$$L_1 k_1 + L_2 k_2 = K. \tag{9}$$

With this specification of the supply side of the model the factor prices are determined.

As mentioned earlier, we assume a small country because the assumption of a large country implies random product prices, the ultimate value of which is known only after α is resolved. However, producers make their decisions on the basis of already known product prices. This conflict does not arise in the case of a small country, where product prices are given from the world markets of X_1 and X_2.

Having laid out our theoretical model above, we go on to analyze different general equilibrium, tax incidence results under uncertainty in Section 2.

2. UNCERTAINTY AND TAX INCIDENCE

In order to expedite consideration of the incidence and differential incidence of taxes in perfect competition, the relationships between factor prices in the different sectors, and between producers' and consumers' product prices under different taxes are presented below.

Partial factor tax. (i) In X_1 on capital and labor, respectively,

$$r = r_2 = (1 - t_{K_1}) r_1 ,$$

$$w = w_2 = (1 - t_{L_1}) w_1 .$$

(ii) In X_2 on capital and labor, respectively,

$$r = r_1 = (1 - t_{K_2}) r_2 ,$$

$$w = w_1 = (1 - t_{L_2}) w_2 .$$

Partial commodity tax. (i) On X_1: $p_1 = (1 - t_1) p^1$.

(ii) on X_2: $p_2 = (1 - t_2) p^2$.

General factor tax. (i) On capital: $r = (1 - t_K) r_1 = (1 - t_K) r_2$.

(ii) On labor: $w = (1 - t_L) w_1 = (1 - t_L) w_2$.

General income tax.

$$r = (1 - t_Y) r_1 = (1 - t_Y) r_2 ,$$

and

$$w = (1 - t_Y) w_1 = (1 - t_Y) w_2 .$$

General sales tax.

$$p_1 = (1 - t_S) p^1$$

and

$$p_2 = (1 - t_S) p^2,$$

where r and w are the net rates of return to capital and labor, respectively, t_{ij} denotes a partial factor tax on the ith factor in the jth sector, t_j $(j = 1, 2)$ denotes a partial commodity tax on the jth commodity, t_i $(i = L, K)$ denotes a general factor tax on the ith factor, and t_Y and t_S denote a general income and sales tax, respectively.

For convenience and clarity we divide the rest of this section into three subsections, studying the incidence and differential incidence of partial taxes in the certain sector X_1, partial taxes in the uncertain sector X_2, and general taxes, respectively.

(a) *Partial Taxes in the Certain Sector*, X_1

Suppose capital in the certain sector X_1 is taxed at rate t_{K_1}. Then, a perfectly competitive capital market implies that $r = r_2 = (1 - t_{K_1}) r_1$, i.e., the net rate of return to capital, r, is equal in both sectors. Substituting (6) and (7) into (3) and (4), respectively, yields

$$E[U_2'(\Pi_2)\{p_2\alpha(f_2 - k_2f_2') - p_1(f_1 - k_1f_1')\}] = 0 \qquad (10$$

and

$$E[U_2'(\Pi_2)\{p_2\alpha f_2 - (1 - t_{K_1})\,p_1f_1'\}] = 0. \qquad (11)$$

Differentiating (8) and (9) with respect to t_{K_1} yields

$$\frac{dL_2}{dt_{K_1}} = -\frac{1}{k_2 - k_1}\left(L_1\frac{dk_1}{dt_{K_1}} + L_2\frac{dk_2}{dt_{K_1}}\right). \qquad (12)$$

Differentiating (2), (10), and (11) with respect to t_{K_1} and using (12) yields the following system of equations, as derived in Appendix A.

$$\begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ k_2 - k_1 & L_2 & L_1 \end{bmatrix} \begin{bmatrix} (dL_2/dt_{K_1}) \\ (dk_1/dt_{K_1}) \\ (dk_2/dt_{K_1}) \end{bmatrix} = \begin{bmatrix} H_1(t_{K_1}) \\ H_2(t_{K_1}) \\ 0 \end{bmatrix}, \qquad (13)$$

where

$$\alpha_1 = \frac{w}{r}\alpha_2 = E\left[U_2''(\Pi_2)(\alpha p_2F_{K_2} - r)^2\frac{w}{r}\left(\frac{w}{r} + k_2\right)\right],$$

$$\beta_1 = E\left[U_2''(\Pi_2)(\alpha p_2F_{K_2} - r)^2 L_2\frac{w}{r} - p_2k_2f_2''\alpha U_2'(\Pi_2)\right],$$

$$\beta_2 = E\left[U_2''(\Pi_2)(\alpha p_2F_{K_2} - r)^2 L_2 + p_2f_2''\alpha U_2'(\Pi_2)\right],$$

$$\gamma_1 = E\left[U_2''(\Pi_2)(\alpha p_2F_{K_2} - r)\frac{w}{r}(k_1 - k_2)\,p_1f_1''L_2 + p_1k_1f_1''U_2'(\Pi_2)\right],$$

$$\gamma_2 = E\left[U_2''(\Pi_2)(\alpha p_2F_{K_2} - r)(k_1 - k_2)\,p_1f_1''L_2 - p_1f_1''U_2'(\Pi_2)\right],$$

$$H_1(t_{K_1}) = -rE[U_2''(\Pi_2)(\alpha p_2F_{K_2} - r)]\frac{w}{r}L_2k_2,$$

and

$$H_2(t_{K_1}) = -rE[U_2''(\Pi_2)(\alpha p_2F_{K_2} - r)]\,L_2k_2 - rE[U_2'(\Pi_2)].$$

It can be shown that D, the denominator of the system, is positive if we have nonincreasing absolute risk aversion and if the tax is evaluated at zero revenues, as in Harberger [5] and Mieszkowski [9].[1] Throughout the following discussion we assume $D > 0$.

Solving for

$$\frac{dk_1}{dt_{K_1}} = \frac{H_1(t_{K_1})\{\alpha_2 L_2 - (k_2 - k_1)\beta_2\} - H_2(t_{K_1})\{\alpha_1 L_2 - (k_2 - k_1)\beta_1\}}{D}$$

and using $dr/dt_{K_1} = -r + p_1 f_1''(dk_1/dt_{K_1})$[2], we obtain

$$\frac{dr}{dt_{K_1}} = \frac{rk_1}{k_2 - k_1}\left[1 - \frac{\phi(t_{K_1})}{D}\right], \tag{14}$$

where

$$\phi(t_{K_1}) = \left(\frac{w}{r} + k_2\right) L_2 \left(\frac{w}{r} + k_1\right) p_1 f_1'' E[U_2'(\Pi_2)]\, E[U_2''(\Pi_2)(\alpha p_2 F_{K_2} - r)^2]$$
$$+ \frac{k_2}{k_1} L_1 p_2 \frac{f_2 f_2''}{f_2'} \left(\frac{w}{r} + k_2\right) E[\alpha U_2'(\Pi_2)]\, E[U_2''(\Pi_2)(\alpha p_2 F_{K_2} - r)^2].$$

It may be noted that $\phi(t_{K_1}) = 0$ in the case of risk neutrality and we have the same effect on r of the partial capital tax as in the certainty case. In the risk averse case $\phi(t_{K_1}) > 0$, and since $D > 0$,

$$\left|\frac{dr}{dt_{K_1}}\right|\Bigg|_{U_2''=0} > \left|\frac{dr}{dt_{K_1}}\right|\Bigg|_{U_2''<0},$$

i.e., the effect on r of a partial capital tax is less pronounced under risk aversion than under risk neutrality.

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[1] $$D = p_1 p_2 f_1'' f_2''(k_2 - k_1)^2 E[U_2'(\Pi_2)] E[\alpha U_2'(\Pi_2)]$$
$$+ p_1 p_2 f_1'' f_2'' \frac{f_2}{f_2'} L_2(k_2 - k_1)^2 E[U_2''(\Pi_2)(\alpha p_2 F_{K_2} - r)] E[\alpha U_2'(\Pi_2)]$$
$$+ \left(\frac{w}{r} + k_2\right) L_1 p_2 \frac{f_2 f_2''}{f_2'} E[U_2''(\alpha p_2 F_{K_2} - r)^2] E[\alpha U_2'(\Pi_2)]$$
$$+ \left(\frac{w}{r} + k_1\right) L_2 p_1 \frac{f_1 f_1''}{f_1'} E[U_2'(\Pi_2)].$$

Given $U''(\Pi_2) < 0$, all the expressions are positive except the second expression. In Appendix B it is shown that $E[U''(\Pi_2)(\alpha p_2 F_{K_2} - r)] > 0$ under nonincreasing absolute risk aversion.

[2] Since $r = (1 - t_{K_1}) p_1 f_1'$.

Similarly, a partial tax on labor in the certain sector X_1 implies $w = w_2 = (1 - t_{L_1})\, w_1$ given perfectly competitive labor markets. In this case,[3]

$$\frac{dr}{dt_{L_1}} = \frac{w}{k_2 - k_1}\left[1 - \frac{\phi(t_{L_1})}{D}\right], \tag{15}$$

where

$$\phi(t_{L_1}) = \left(\frac{w}{r} + k_1\right)\left(\frac{w}{r} + k_2\right) L_2 p_1 f_1'' E[U_2'(\Pi_2)]\, E[U_2''(\Pi_2)(\alpha p_2 F_{K_2} - r)^2]$$
$$+ \left(\frac{w}{r} + k_2\right) L_1 p_2 \frac{f_2 f_2''}{f_2'} E[\alpha U_2'(\Pi_2)]\, E[U_2''(\Pi_2)(\alpha p_2 F_{K_2} - r)^2]$$
$$\gtrless 0 \qquad \text{according as } U_2''(\Pi_2) \lessgtr 0,$$

i.e., the effect on r of a partial labor tax is also less pronounced under risk aversion than under risk neutrality.

Finally, in the case of a partial commodity tax on X_1, $p_1 = (1 - t_1)\, p^1$, where p^1 is the given world price as noted earlier. Here[4]

$$\frac{dr}{dt_1} = \frac{w + rk_1}{k_2 - k_1}\left[1 - \frac{\phi(t_1)}{D}\right], \tag{16}$$

where

$$\phi(t_1) = \frac{w + rk_2}{w + rk_1}\left(\frac{w}{r} + k_2\right) L_1 p_2 \frac{f_2 f_2''}{f_2'} E[\alpha U_2'(\Pi_2)]\, E[U_2''(\Pi_2)(\alpha p_2 F_{K_2} - r)^2]$$
$$+ \left(\frac{w}{r} + k_1\right) L_2 p_1 \frac{f_1 f_1''}{f_1'} E[U_2'(\Pi_2)]$$
$$\gtrless 0 \qquad \text{according as } U_2''(\Pi_2) \lessgtr 0.$$

[3] $$\frac{dk_1}{dt_{L_1}} = \frac{H_1(t_{L_1})\{\alpha_2 L_2 - (k_2 - k_1)\beta_2\} - H_2(t_{L_1})\{\alpha_1 L_2 - (k_2 - k_1)\beta_1\}}{D},$$

where

$$H_1(t_{L_1}) = -wE[U_2'(\Pi_2)] - wL_2 E[U_2''(\Pi_2)(\alpha p_2 F_{K_2} - r)][w/r],$$

and

$$H_2(t_{L_1}) = -wL_2 E[U_2''(\Pi_2)(\alpha p_2 F_{K_2} - r)].$$

Also note

$$(dr/dt_{L_1}) = p_1 f_1''(dk_1/dt_{L_1}).$$

[4] In this case,

$$H_1(t_1) = (w/r)H_2(t_1) = -wE[U_2'(\Pi_2)] - (wL_2 + rk_2 L_2)E[U_2''(\Pi_2)(\alpha p_2 F_{K_2} - r)][w/r].$$

Comparing (14), (15), and (16), we get the following results. Putting $dt_{K_1} = dt_{L_1} = dt_1$, we get $(dr/dt_{K_1}) + (dr/dt_{L_1}) = dr/dt_1$, i.e., the effect of the two equal rate partial factor taxes in X_1 is equivalent to a commodity tax on X_1 at the same rate. This certainty result on tax incidence [9] thus carries over in the general case of uncertainty, regardless of the producers' attitude toward risk.

For partial, equal-yield taxes, $rK_1dt_{K_1} = wL_1dt_{L_1} = pL_1f_1dt_1$. Substituting $dt_{K_1} = [p_1L_1f_1/(rK_1)]\,dt_1$ in (14), $dt_{L_1} = [pL_1f_1/(wL_1)]\,dt_1$ in (15) and rewriting (16), we get

$$dr(t_{K_1}) = \frac{p_1 f_1}{k_2 - k_1}\left[1 - \frac{\phi(t_{K_1})}{D}\right] dt_1,$$

$$dr(t_{L_1}) = \frac{p_1 f_1}{k_2 - k_1}\left[1 - \frac{\phi(t_{L_1})}{D}\right] dt_1,$$

and,

$$dr(t_1) = \frac{p_1 f_1}{k_2 - k_1}\left[1 - \frac{\phi(t_1)}{D}\right] dt_1,$$

respectively. With risk neutrality all these equal-yield, partial taxes have the same effect on factor prices, as in the small country, certainty case. With risk aversion, $\phi(t_{K_1}) \gtrless \phi(t_1) \gtrless \phi(t_{L_1})$ according as $k_2 \gtrless k_1$. This implies $dr(t_{K_1}) < dr(t_1) < dr(t_{L_1})$ with all of them $\gtrless 0$ according as $k_2 \gtrless k_1$. Thus capital is worse off under the partial capital tax in X_1 than under the equal-yield, partial commodity tax on X_1, and worse under the latter than an equal-yield, partial labor tax in X_1, with the reverse holding for labor.

(b) *Partial Taxes in the Uncertain Sector*, X_2

With partial factor and commodity taxes in the uncertain sector X_2, results derived on the incidence of partial taxes in the certain sector X_1 may no longer hold. In this case,

$$\frac{dr}{dt_{K_2}} = -\frac{rk_2}{k_2 - k_1}\left[1 - \frac{k_1}{k_2}\frac{\phi(t_{K_1})}{D}\right], \tag{17}$$

$$\frac{dr}{dt_{L_2}} = -\frac{w}{k_2 - k_1}\left[1 - \frac{\phi(t_{L_1})}{D}\right], \tag{18}$$

and

$$\frac{dr}{dt_2} = -\frac{w + rk_2}{k_2 - k_1}\left[1 - \frac{w + rk_1}{w + rk_2}\frac{\phi(t_1)}{D} + \frac{(k_2 - k_1)^2}{D}\left\{p_1 p_2 f_1'' \frac{f_2 f_2''}{f_2'} L_2 E[\alpha U_2'(\Pi_2)]\, E[U_2''(\Pi_2)(\alpha p_2 F_{K_2} - r)^2\right\}\right]. \tag{19}$$

We may note, first, that equal-rate partial factor taxes are not equivalent to an equal-rate commodity tax on X_2 under risk aversion. In fact, the relatively intensive factor is worse off under the former than under the latter. This is because the differential effect on expected output of a commodity tax in the uncertain sector X_2, compared to equal-rate, partial factor taxes, is not zero. This is in contrast to the effects in the certain sector X_1, where the differential effect on output is zero, hence yielding the equivalence between partial factor taxes and a commodity tax in the certain sector, as noted above. However, exactly as in the case of partial taxes in the certain sector in the case of partial taxes in the uncertain sector, the incidence effects are reduced under risk aversion compared to risk neutrality.

For the consideration of equal-yield, partial taxes in the uncertain sector X_2, it is meaningful to talk only of expected revenues from the partial commodity tax on X_2, i.e., $E[\alpha p_2 L_2 f_2]\, dt_2$. The condition for equal-yield commodity and factor taxes in this sector is then $rK_2 dt_{K_2} = \mu p_2 L_2 f_2 dt_2 = wL_2 dt_{L_2}$. Under risk neutrality, the effect of equal yield, partial taxes is the same Given risk aversion, it can be shown that capital fares worse under a partial capital tax in X_2 than under a partial labor tax. However, it is no longer true that capital bears a greater burden under a partial capital tax than under a partial commodity tax on X_2 unless X_2 is capital intensive. Also, capital need not bear a greater burden under a partial commodity tax on X_2 than under a partial labor tax in X_2, unless X_2 is labor intensive.[5]

(c) *General Factor, Income, and Sales Taxes*

The well-known result that a general factor tax is borne entirely by the taxed factor under certainty continues to hold under uncertainty, provided there is constant absolute risk aversion. With a general factor tax on capital,[6]

$$\frac{dr}{dt_K} = \frac{r}{D} - \frac{p_1 p_2 f_1''}{D} \frac{f_2 f_2''}{f_2'} (k_2 - k_1)\, rK_2 E[\alpha U_2'(\Pi_2)]\, E[U_2''(\Pi_2)(\alpha p_2 F_{K_2} - r)] \tag{20}$$

and

$$\frac{dw}{dt_K} = \frac{p_1 p_2 f_1''}{D} \frac{f_2 f_2''}{f_2'} k_1(k_2 - k_1)\, rK_2 E[\alpha U_2'(\Pi_2)]\, E[U_2''(\Pi_2)(\alpha p_2 F_{K_2} - r)]. \tag{21}$$

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[5] From the expressions of (dr/dt_{K_2}), (dr/dt_{L_2}), and (dr/dt_2) it follows that $dr(t_{K_2}) < dr(t_{L_2})$, $dr(t_{K_2}) < dr(t_2)$ if $k_2 > k_1$ and $dr(t_2) < dr(t_{L_2})$ if $k_1 > k_2$, given $U_2''(\Pi_2) < 0$.

[6] Here $H_1(t_K) = (w/r)H_2(t_K) = wK_2 E[U_2''(\Pi_2)(\alpha p_2 F_{K_2} - r)]$. It should be noted that in general $(dr/dt_{K_1}) + (dr/dt_{K_2}) \neq (dr/dt_K)$. This unusual result is explicable since each of these taxes has been introduced into the model separately. This implies that the net rate of return to capital, with a partial tax on capital in X_1, is a different function, say $r(t_{K_1})$, from the net rate of return to capital, say $r(t_{K_2})$, under a partial tax on capital in X_2. Further, $r(t_{K_1}) + r(t_{K_2})$ need not be the same as $r(t_K)$, the net rate of return on capital as a function of a general capital tax.

It may be shown that $E[U_2''(\Pi_2)(\alpha F_{K_2} - r)] \gtreqless 0$ according as $R_A'(\Pi_2) \lesseqgtr 0$, as shown in Appendix B where R_A is the familiar risk-aversion function. It follows that if the uncertain sector X_2 is capital intensive, i.e., $k_2 > k_1$,

$$dr/dt_K \lesseqgtr -r, \qquad \text{according as } R_A'(\Pi_2) \lesseqgtr 0$$

and

$$dw/dt_K \gtreqless 0, \qquad \text{according as } R_A'(\Pi_2) \lesseqgtr 0,$$

with the reverse holding for $k_1 > k_2$. It follows that under the plausible assumption of decreasing absolute risk aversion, i.e., $R_A' < 0$, if the more (less) intensive factor of the uncertain sector is taxed, it bears a greater (less) burden than under certainty or constant absolute risk aversion, while the untaxed, less (more) intensive factor gains (loses).

We should further note here that capital bears 100% of the burden of a general capital tax if the ratio of gross income of capital, i.e., net income plus tax revenues, to labor income remains unchanged, despite the imposition of the tax, i.e.,

$$d\left[\frac{rk(1+t_K)}{wL}\right] = 0 \qquad \text{or} \qquad \frac{dr}{dt_K} = -r + \frac{r}{w}\frac{dw}{dt_K}.$$

In the certainty analysis this condition is met since $dr/dt_K = -r$ and $dw/dt_K = 0$. Given product prices and that the return to labor remains unchanged, the real income of labor is unchanged, and capital must bear the entire burden of the tax. However, the above condition need not hold under uncertainty.

Under certainty, the replacement of a general labor tax by a general capital tax of equal yield results in a shifting of the entire burden of the tax from labor to capital. Under uncertainty, while it is no longer true that a general factor tax is precisely borne by the taxed factor, this result on the differential incidence of equal-yield, general factor taxes continues to hold under uncertainty, regardless of the producers' attitude towards risk, in the form that the replacement of a general labor tax by a general capital tax reverses the burdens borne by labor and capital, capital losing by the value of the tax and labor gaining by the same amount. A general capital tax is of equal yield to a general labor tax if $dt_L = (rK_2/wL_2)\,dt_K$. Substituting this into

$$\frac{dr}{dt_L} = -p_1 p_2 f_1'' \frac{f_2 f_2''}{f_2'}(k_1 - k_2)\, wL_2 E[\alpha U_2'(\Pi_2)]\, E[U_2''(\Pi_2)(\alpha p_2 F_{K_2} - r)] \qquad (22)$$

and subtracting (22) from (20) yields

$$\frac{dr(t_K)}{dt_K} - \frac{dr(t_L)}{dt_K} = -r.$$

The certainty result is a special case of this, since $dr/dt_K = -r$ and $dr/dt_L = 0$.

The certainty result that a general income tax is borne by both factors in proportion to their initial shares in income also continues to hold under uncertainty provided there is constant absolute risk aversion. With a general income tax on the earnings of the two factors of production,[7],

$$\frac{dr}{dt_Y} = -r - p_1 p_2 f_1'' \frac{f_2 f_2''}{f_2'} (k_2 - k_1)(w + rk_2)\, L_2 E[\alpha U_2'(\Pi_2)]\, E[U_2''(\Pi_2) \times (\alpha p_2 F_{K_2} - r)] \tag{23}$$

and

$$\frac{dw}{dt_Y} = -w + p_1 p_2 k_1 f_1'' \frac{f_2 f_2''}{f_2'} (k_2 - k_1)(w + rk_2)\, L_2 E[\alpha U_2'(\Pi_2)]\, E[U_2''(\Pi_2) \times (\alpha p_2 F_{K_2} - r)]. \tag{24}$$

Expressions (23) and (24) reduce to $-r$ and $-w$, respectively, under constant absolute risk aversion, i.e., $R_A'(\Pi_2) = 0$. This implies that the factors bear the income tax in proportion to their initial shares in income. This is because the condition for equal proportional burdens, viz.,

$$d\left[\frac{(1 - t_Y)\, rK}{(1 - t_Y)\, wL}\right] = d\left[\frac{rK}{wL}\right] = 0$$

implies $dr/dt_Y = (r/w)(dw/dt_Y)$, which is satisfied by $dr/dt_Y = -r$ and $dw/dt_Y = -w$. With declining absolute risk aversion, the more (less) intensive factor in the uncertain sector bears a greater (less) than proportionate burden of the income tax, the reverse holding for increasing absolute risk aversion. Therefore, with $k_2 > k_1$, i.e., the uncertain sector X_2 is capital intensive, we have

$$\frac{dr}{dt_Y} \gtreqless -r, \qquad \frac{dw}{dt_Y} \lesseqgtr -w, \qquad \text{according as } R_A' \gtreqless 0.$$

In the analysis of a general sales tax, the certainty result that the tax is borne by the two factors in proportion to their initial shares in income is no longer true under uncertainty, even under constant absolute risk aversion. In this case,[8]

$$\frac{dr}{dt_s} = -r - p_1 p_2 f_1'' \frac{f_2 f_2''}{f_2'} \left(\frac{w}{r} + k_1\right)(k_2 - k_1)\, L_2 E[\alpha U_2'(\Pi_2)]\, E[U_2''(\Pi_2) \times (\alpha F_{K_2} - r)^2] \tag{25}$$

---

[7] Since $H_1(t_Y) = (w/r)H_2(t_Y) = (w/r)L_2(w + rk_2)E[U''(\alpha p_2 F_{K_2} - r)]$.

[8] Since $H_1(t_S) = (w/r)H_2(t_S) = (w/r)L_2(w + rk_2)E[U''(\alpha p_2 F_{K_2} - r)^2]$.

and

$$\frac{dw}{dt_S} = -w \cdot p_1 p_2 k_1 f_1'' \frac{f_2 f_2''}{f_2'} \left(\frac{w}{r} + k_1\right)(k_2 - k_1)\, L_2 E[\alpha U_2'(\Pi_2)]\, E[U_2''(\Pi_2)$$
$$\times (\alpha F_{K_2} - r)^2]. \tag{26}$$

From (25) and (26) we find that the certainty result holds under risk neutrality. But under risk aversion, the more (less) intensive factor in the uncertain sector bears a less (more) than proportionate burden than if the producers were risk neutral, e.g., given $k_2 > k_1$,

$$\frac{dr}{dt_S} \gtrless -r, \qquad \frac{dw}{dt_S} \lessgtr -w, \qquad \text{according as } U''(\Pi_2) \lessgtr 0.$$

Comparing the incidence of a general income tax to a general sales tax we may note that under certainty and risk neutrality, the replacement of a general income tax by an equal-yield, general sales tax does not alter factor shares in income. Under decreasing absolute risk aversion, however, the same replacement makes the more (less) intensive factor in the uncertain sector better (worse) off than before.

Having analyzed the incidence and differential incidence of several partial and general taxes in a general equilibrium model under uncertainty, we give our conclusions in Section 3.

3. CONCLUSION

The effect of partial taxes on factor prices is less pronounced if producers are risk averse than if they are risk neutral. Further, the differential incidence of factor taxes in the certain sector with an equal rate commodity tax in that sector is zero, though in the uncertain sector the relatively intensive factor gains. In the certain sector, the replacement of a commodity tax by an equal-yield, partial factor tax makes the taxed (untaxed) factor worse (better) off under risk aversion, though the effect on factor price is the same under risk neutrality. In the uncertain sector, it remains true that the differential incidence on capital (labor) by replacing a partial capital tax with an equal-yield, partial labor tax is positive (negative) under risk aversion, the certainty result continuing to hold under risk neutrality. But the differential incidence of partial factor taxes on the one hand, and an equal yield, partial commodity tax on the other, turns on relative factor intensities in the two sectors, in the case of risk aversion, the certainty result continuing to hold under risk neutrality.

Given constant absolute risk aversion, the effects of a general factor tax and an income tax are the same as under certainty. Given declining (increasing) absolute risk aversion, the relatively intensive factor in the uncertain sector is worse (better) off under these taxes than under constant absolute risk aversion. The differential incidence of the two general factor taxes is the same as in the certainty case, regardless of the attitude towards risk. In the case of a general sales tax, the relatively intensive factor in the uncertain sector bears a smaller burden in the risk averse than in the risk neutral case. These results indicate that the effect of a replacement of a general income tax by an equal-yield, general sales tax is to make the more (less) intensive factor of the uncertain sector better (worse) off, provided risk aversion is decreasing. However, it should be noted that all of these results were obtained for the small country case and with uncertainty in one sector.

APPENDIX A

Noting that for a partial factor tax on capital in X_1, $r = r_2 = (1 - t_{K_1}) r_1$, and using (6) and (7),

$$\Pi_2 = L_2[\alpha p_2 f_2(k_2) - (1 - t_{K_1}) p_1 f_1'(k_1) k_2 - p_1(f_1 - k_1 f_1')]. \tag{A.1}$$

Differentiating (10) and (11) with respect to t_{K_1} and evaluating the results at $t_{K_1} = 0$, yields

$$\frac{d\Pi_2}{dt_{K_1}} = \frac{\Pi_2}{L_2}\frac{dL_2}{dt_{K_1}} + L_2(\alpha p_2 F_{K_2} - r)\frac{dk_2}{dt_{K_1}} - L_2 p_1 f_1''(k_2 - k_1)\frac{dk_1}{dt_{K_1}} + rL_2k_2, \tag{A.2}$$

$$E\left[U_2''(\Pi_2)(\alpha p_2 F_{L_2} - w)\frac{d\Pi_2}{dt_{K_1}}\right] - E[U_2'(\Pi_2)\,\alpha p_2 k_2 f_2'']\frac{dk_2}{dt_{K_1}}$$
$$+ E[U_2'(\Pi_2)\,p_1 k_1 f_1'']\frac{dk_1}{dt_{K_1}} = 0, \tag{A.3}$$

and

$$E\left[U_2''(\Pi_2)(\alpha p_2 F_{K_2} - r)\frac{d\Pi_2}{dt_{K_1}}\right] + E[U'(\Pi_2)\,\alpha p_2 k_2 f'']\frac{dk_2}{dt_{K_1}}$$
$$- E[U_2'(\Pi_2)\,p_1 f_1'']\frac{dk_1}{dt_{K_1}} = rE[U_2'(\Pi_2)]. \tag{A.4}$$

Substituting the value of $d\Pi_2/dt_{K_1}$ from (A.2) into (A.3) and (A.4) gives us the first two equations of our three-equation system (13). The third equation is given by (12) in the text.

APPENDIX B

We show $E[U_2''(\Pi_2)(\alpha p_2 F_{K_2} - r)] \gtreqless 0$ according as $R_A' \lesseqgtr 0$: For $\alpha > r/p_2F_{K_2}$, $\Pi_2 > 0$ since $\Pi_2 = L_2[\alpha p_2 F_{K_2} - r][(w/r) + k_2]$. Then, given $R_A' < 0$, i.e., decreasing absolute risk aversion where $R_A(\Pi_2) = -U_2''(\Pi_2)/U_2'(\Pi_2)]$, we have

$$-\frac{U_2''(\Pi_2)}{U_2'(\Pi_2)} < R_A[\Pi_2 = 0]$$

and

$$-(\alpha p_2 F_{K_2} - r)\, U_2''(\Pi_2) < (\alpha p_2 F_{K_2} - r)\, U_2'(\Pi_2)\, R_A[\Pi_2 = 0]. \qquad \text{(B.1)}$$

For $\alpha < (r/p_2F_{K_2})$, $\Pi_2 < 0$. Then

$$-\frac{U_2''(\Pi_2)}{U_2'(\Pi_2)} > R_A[\Pi_2 = 0],$$

and

$$-(\alpha p_2 F_{K_2} - r)\, U_2''(\Pi_2) < (\alpha p_2 F_{K_2} - r)\, U_2'(\Pi_2)\, R_A[\Pi_2 = 0]. \qquad \text{(B.2)}$$

Adding (B.1) and (B.2), taking expectations of both sides, and observing (4) yields

$$E[U_2''(\Pi_2)(\alpha p_2 F_{K_2} - r)] > 0.$$

The results for $R_A' \geq 0$ follow similarly.

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Existence of Equilibria in Economies with a Local Public Good

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1. INTRODUCTION

In a seminal article, Tiebout [7] suggested that the benefits of most goods provided by the public sector are limited to those consumers who reside in a relatively small geographical area (which he calls a community) and do not extend over the entire population. This "localness" makes it possible for consumers to choose among many different communities, each offering different levels of local public goods and charging different "prices of residency" (taxes). Thus, in a sense, the communities are placed in a market where they must compete for residents. Tiebout draws an analogy between this "marketplace of communities" in the public sector and a "marketplace of goods" in the private sector to argue that this decentralized community choice process will result in the efficient provision of local public goods [7].

While the most obvious criticism of Tiebout's argument is its lack of rigor, other, more fundamental objections may be raised. In particular, his analysis is crucially dependent upon three special assumptions. First, he assumes that there are enough different communities so as to give consumers the opportunity to choose from a complete spectrum of technologically feasible public good/tax packages. (Consequently, each community's budgetary policy should command the unanimous support of its residents, and an intra-community decision-making process is unnecessary.) Second, he postulates that local public goods are not pure public goods in the traditional sense: The per capita cost of an arbitrary level of a local public good is assumed to be a U-shaped function of community size. Furthermore, his analysis specifies that each community providing a given level of a local public good will be populated to the point where increasing returns to scale with respect to community size are exhausted (i.e., the point of minimum per capita cost).

In this paper, we shall develop a formal model in which alternative

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assumptions concerning the spectrum of available communities and the type of technology are made. We shall assume that due to a scarcity of community sites each consumer must choose among a finite number of communities. As a result, intracommunity disagreement over budgetary policy will typically occur. A majority rule voting process is used to resolve such differences. Also, the local public good will be a Samuelson pure public good within each community, and consequently increasing returns to scale with respect to community size are never exhausted. An equilibrium is said to exist whenever the consumers are partitioned into communities such that each community is in "voting equilibrium" and no consumer would become better off by migrating to another community. The standard restrictions usually placed upon consumer preferences and production do not guarantee the existence of an equilibrium with an arbitrary number of communities. In order to ensure existence, some restriction must be placed upon relative consumer preferences. To argue these points the body of the paper is organized into three sections. In Section 2 we explain and develop the model rather informally. Section 3 introduces our relative preference assumptions and outlines an existence proof. Last, in Section 4 the role played by these "special" assumptions in guaranteeing existence is examined.

2. THE MODEL

Since this section is designed to be an informal discussion of the model, all formal definitions will appear later. Our economy consists of a continuum of consumers on the interval [0, 1] and two goods, one local public and one private. The consumers are partitioned into M communities (where M is a positive integer) which are denoted by the integers 1, 2,..., M. Each com-

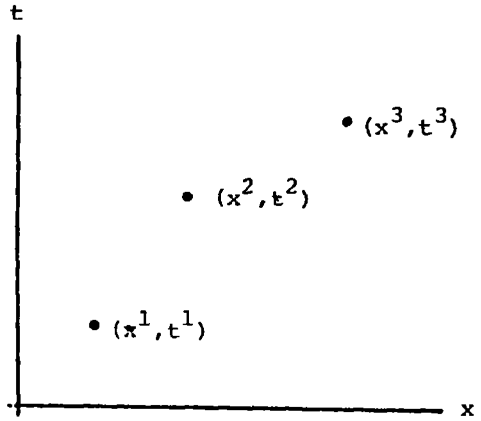

FIGURE 1

munity produces a certain level of the local public good for its residents and finances production costs by levying a proportional income tax upon these residents. Although other tax schemes could be studied, this one proved the most tractable. We can represent the ith community by the ordered pair (x^i, t^i), where x^i is the level of the local public good produced in community i and t^i is the tax rate levied by community i. Therefore, each community can be denoted by a point in (x, t) space (see Fig. 1). (It is required that in each community $x^i \geq 0$, $t^i \leq 1$.)

An equilibrium exists whenever the consumers are partitioned into the communities such that (1) no consumer desires to switch from the community in which he resides, and (2) the (x, t) offered by each community is a community voting equilibrium. The notion of switching is very natural: An individual desires to switch from one community to another whenever he would be better off in the second community.[1] Throughout this analysis we will assume that each individual takes the levels of the local public good and tax rate offered by his and other communities as fixed; that is, he does not account for the effect of consumer migration upon the (x, t)'s offered by the communities.

In order for community i to be in voting equilibrium, (x^i, t^i) must satisfy two conditions. First, the community must balance its budget. We shall assume that within each community the pure public good has a linear cost function (note that this assumption can be relaxed; see Footnote 3); i.e., production costs in community i equal cx^i. Let

$A^i =$ the set of consumers who reside in community i,

and

${}^n y =$ the income of consumer n.

Thus, for a balanced budget,

$$cx^i = \int_{A^i} t^i\, {}^n y\, dn.$$

This community budget constraint can be rewritten as $cx^i = t^i \bar{y}^i$, where

$$\bar{y}^i = \int_{A^i} {}^n y\, dn.$$

The second condition required is that (x^i, t^i) result from a binary majority rule voting process. That is, over all (x, t) on community i's budget constraint, at least half of the consumers who reside in community i do not desire less x than x^i and at least half do not desire more than x^i.

---

[1] In the formal definition which appears in Section 3, switching is assumed to occur even when individuals are indifferent, in order to rule out the possibility of trivial equilibria involving two or more communities offering the same (x, t).

It is instructive to illustrate an equilibrium graphically. To do this we must be able to represent consumer preferences in (x, t) space. Let ${}^nu = {}^nu(x, z)$, where

$$
\begin{aligned}
{}^nu &= \text{utility of consumer } n,\\
x &= \text{public good},\\
z &= \text{private good}.
\end{aligned}
$$

We can determine utility as a function of x and t by solving the problem

$$
\max_z \ {}^nu = {}^nu(x, z),
$$
$$
\text{s.t. } {}^ny \geq t\,{}^ny + z.
$$

We shall assume monotonicity; therefore, the solution will be ${}^nu = {}^nu(x, (1 - t)\,{}^ny)$. The shape of consumer n's indifference curves in (x, t) space can be determined by totally differentiating ${}^nu$ with respect to x and t:

$$
0 = {}^nu_1\,dx - {}^ny\,{}^nu_2\,dt,
$$

$$
\frac{dt}{dx} = \frac{{}^nu_1}{{}^ny\,{}^nu_2},
$$

$$
\frac{d^2t}{dx^2} = \frac{{}^nu_2^2\,{}^nu_{11} - 2\,{}^nu_1\,{}^nu_2\,{}^nu_{12} + {}^nu_1^2\,{}^nu_{22}}{{}^ny\,{}^nu_2^3}.
$$

We shall assume that ${}^ny > 0$; hence, $dt/dx > 0$. Also, we shall assume that ${}^nu$ is strictly quasi-concave; hence,

$$
{}^nu_2^2\,{}^nu_{11} - 2\,{}^nu_1\,{}^nu_2\,{}^nu_{12} + {}^nu_1^2\,{}^nu_{22} < 0.
$$

Thus, $d^2t/dx^2 < 0$. Therefore, the indifference curves look like the curve shown in Fig. 2.

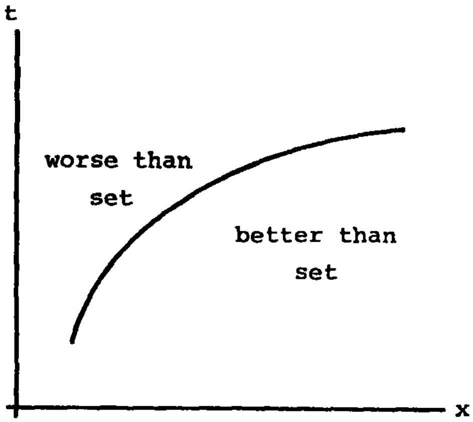

FIGURE 2

We shall first illustrate the conditions needed for (x^i, t^i) to be a community voting equilibrium of community i. Recall that community i must balance its budget; i.e., $cx^i = t^i\bar{y}^i$. Also recall that

$$x^i \geq 0,$$
$$t^i \leq 1.$$

Therefore, $t^i = (c/\bar{y}^i)\,x^i$, where $0 \leq x^i \leq \bar{y}^i/c$. Hence, we can represent community i's budget constraint by the line segment shown in Fig. 3.

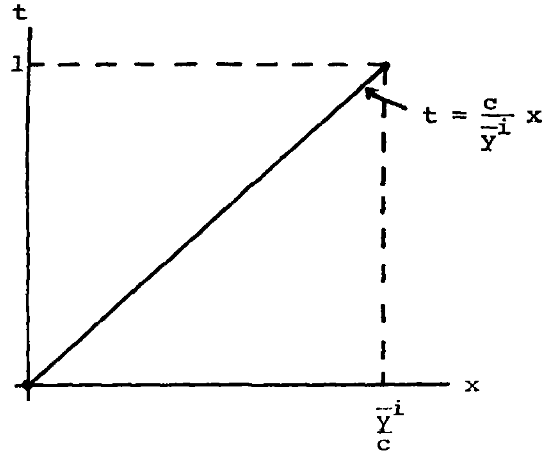

FIGURE 3

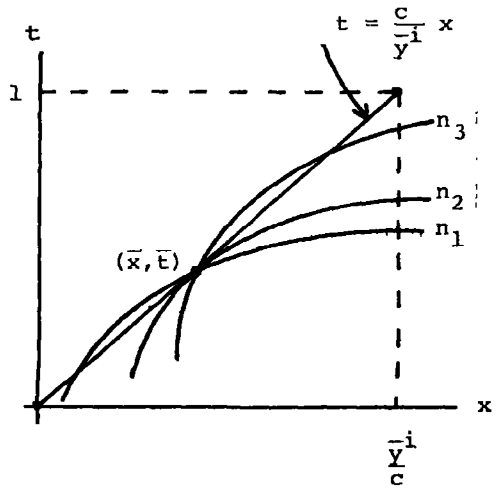

FIGURE 4

For community i to be in voting equilibrium, (x^i, t^i) must not only be a point on this line segment but it must also be a binary majority rule voting equilibrium. To illustrate this second condition consider a point $(\bar{x}, \bar{t})$ which lies on the line segment representing community i's budget constraint (see Fig. 4). Consumer n_1 would be better off at a point on the line $t = (c/\bar{y}^i)x$ below $(\bar{x}, \bar{t})$, and hence, he would desire less x. Consumer n_3 would desire more x and consumer n_2 would desire no change.

Now recall that $({}^n u_1/{}^n y\, {}^n u_2)\,|_{(x,t)}$ is the slope of n's indifference curve at the point (x, t). Thus, at $(\bar{x}, \bar{t})$ consumer n desires

$$\begin{pmatrix}\text{more}\\ \text{same}\\ \text{less}\end{pmatrix} x \qquad \text{whenever} \quad \left(-\frac{{}^n u_1}{{}^n y\, {}^n u_2}\right)\Bigg|_{(\bar{x},\bar{t})} \gtreqless \frac{c}{\bar{y}^i}.$$

Thus, for (x^i, t^i) to be a community voting equilibrium of community i the following two conditions must hold:

(1) (x^i, t^i) must lie on the line $t = (c/\bar{y}^i)x$ to ensure a balanced budget,

and

(2) at least half the consumers in community i must have

$$\left(-\frac{{}^n u_1}{{}^n y\, {}^n u_2}\right)\Bigg|_{(x^i,t^i)} \leq \frac{c}{\bar{y}^i},$$

and at least half must have

$$\left(-\frac{{}^n u_1}{{}^n y\, {}^n u_2}\right)\Bigg|_{(x^i,t^i)} \geq \frac{c}{\bar{y}^i}$$

to ensure majority rule.

To illustrate what is needed to ensure that no consumer desires to switch from his community to another, consider a case in which there are two communities. The two communities are represented in the diagram below by the ordered pairs (x^1, t^1) and (x^2, t^2). In order for no switching to take place, the indifference curves of all the residents of community 1 which pass through (x^1, t^1) must pass "below" (x^2, t^2) and vice versa, as is drawn in Fig. 5. Note that if (x^1, t^1) and (x^2, t^2) are the community voting equilibria of communities 1 and 2, respectively, then the case drawn in Fig. 5 represents an equilibrium.

On the other hand, suppose the indifference curves are not positioned as shown in Fig. 5, but, e.g., as in Fig. 6. In this case, consumer n^* desires to switch from community 2 to community 1 and, hence, the situation drawn is not an equilibrium. If consumer n^* (and a set of consumers of positive measure) does switch, the community voting equilibrium of each community may be affected.

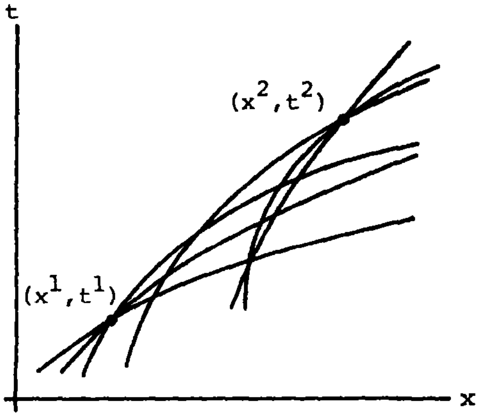

FIGURE 5

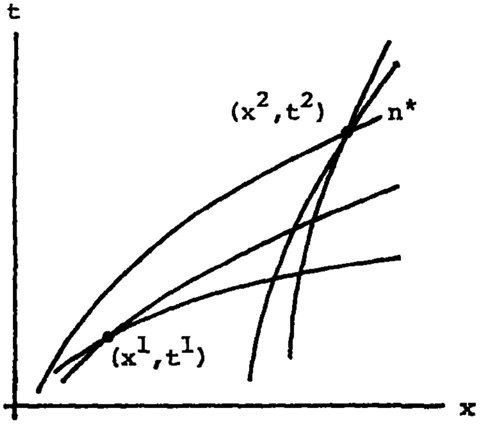

FIGURE 6

Upon inspection of these diagrams it is by no means obvious (and in fact it is not true) that the existence of an equilibrium with two communities can be guaranteed even with the standard restrictions on consumer preferences and production. An example is presented in Section 4 in which an equilibrium does not exist in spite of the fact that the standard conditions are met. However, we shall argue in the following section that existence is guaranteed by the use of two additional assumptions which restrict relative consumer preferences.

3. THE EXISTENCE PROOF[2]

For reference, note that

^nu = utility of the nth consumer,
^ny = income of the nth consumer,
^nx = public good consumed by the nth consumer,
^nz = private good consumed by the nth consumer,
^nt = tax rate faced by the nth consumer,
x^i = public good produced by the ith community,
t^i = tax rate charged by the ith community,
$C(x, N)$ = cost of producing x units of the public good in a community of population N.

We insist that $x^i \geqslant 0$ and $t^i \leqslant 1$. Furthermore, $^nu = {}^nu(^nx, {}^nz)$, where $^nu(^nx, {}^nz)$ is defined over all $^nx \geqslant 0$ and $^nz \geqslant 0$.

We shall now formally define several concepts which were introduced in Section 2.

DEFINITION 1. A good is a local public good if and only if

(1) all residents of the same community consume equal amounts of the good (i.e., it is public within each community);

(2) the production of the good in one community does not affect residents of other communities (i.e., there exist no spillovers between communities).

DEFINITION 2. The measure of the set of consumers A, $\mu(A)$, is the Lebesgue measure.

DEFINITION 3. A set of consumers can form a community if and only if the set has positive measure (i.e., each community must have a positive population).

DEFINITION 4. (x^i, t^i) is a community voting equilibrium of community i if and only if

---

[2] So that the reader may better follow the basic logic of this proof, theorems and arguments are presented less formally than a completely rigorous treatment would require. For example, the Lebesgue measure of the set of consumers who form a community will be called the population of a community. Similarly, we shall use the phrase "no consumers" instead of the phrase "no set of consumers of positive measure."

(1) community i balances its budget;

(2) over all (x, t) on community i's budget constraint, at least half of its residents desire no more x than x^i and at least half desire no less.

DEFINITION 5. Consumer n, who resides in community i, desires to switch to community j $(j \neq i)$ if and only if $(x^j, t^j)\ R_n (x^i, t^i)$.

DEFINITION 6. A partition of the consumers into M communities is an equilibrium if and only if

(1) each community is in voting equilibrium;

(2) no consumers desire to switch.

As previously indicated, we shall restrict our attention to economies with two goods, one private and one public. In addition, we have two other assumptions concerning communities.

ASSUMPTION 1. In each community, revenue is raised by a proportional income tax; i.e., ${}^nt = t^i$ over all n who reside in community i.

ASSUMPTION 2. The cost of producing x units of the local public good in any community is a linear function of x and is unaffected by other variables; i.e., $C(x, N) = cx$.[3]

We shall naturally assume that each consumer maximizes his utility subject to his income. Also, we make the following straightforward assumptions about consumer preferences and endowments.

ASSUMPTION 3. ${}^ny$ is a positive, bounded, measurable function of n over all $n \in [0, 1]$.

ASSUMPTION 4. For each $n \in [0, 1]$,

(1) ${}^nu$ is a continuous function of ${}^nx$ and ${}^nz$,

(2) ${}^nu$ is a strictly quasi-concave function of ${}^nx$ and ${}^nz$ over all ${}^nx > 0$ and ${}^nz > 0$;

(3) ${}^nu_1 > 0$ over all ${}^nx \geqslant 0$, ${}^nz > 0$, ${}^nu_2 > 0$ over all ${}^nx > 0$, ${}^nz \geqslant 0$;

---

[3] The existence proof is not crucially dependent upon this assumption; it is made only to simplify the proof. Assumption 2 can be relaxed to include a much larger class of cost functions: $C(x, N) = \phi(x)\psi(N)$, where (1) average cost with respect to the level of public good produced does not decline; i.e., $(d/dx)\{\phi(x)/x\} \geqslant 0$, (2) average cost with respect to community size does decline; i.e., $(d/dN)(\psi(N)/N) < 0$, (3) $\lim_{N\to 0} (\psi(N)/N) = +\infty$, (4) $\phi(x)$ and $\psi(N)$ are continuous functions of x and N, respectively, over all $x > 0$, $N \geqslant 0$. See [9, pp. 80–85].

(4) If $({}^{n}x, {}^{n}z) \gg (0, 0)$, then ${}^{n}u({}^{n}x, {}^{n}z) > {}^{n}u(0, {}^{n}\bar{z})$ over all ${}^{n}\bar{z} \geq 0$ and ${}^{n}u({}^{n}x, {}^{n}z) > {}^{n}u({}^{n}\bar{x}, 0)$ over all ${}^{n}\bar{x} \geq 0$.

All of these assumptions are rather standard and innocuous. However, they are not sufficient in themselves to guarantee existence. Some restriction must be placed upon the distribution of relative preferences. In the proof below we shall make use of two assumptions concerning the relative slopes of the consumers' indifference curves in (x, t) space (the marginal rate of substitution of the public good for the tax rate). These assumptions enable us to rank individuals so that whenever two are placed in the same community the one with the higher rank will always have the greater marginal rate of substitution.

ASSUMPTION 5. $({}^{n}u_1/{}^{n}y\,{}^{n}u_2)\,|_{(x,t)}$ is an increasing function of n over all (x, t), $x > 0$ and $t < 1$.

ASSUMPTION 6. $({}^{n}u_1/{}^{n}y\,{}^{n}u_2)\,|_{(x,t)}$ is a continuous function of n over all (x, t), $x > 0$ and $t < 1$.[4]

We wish to prove the following theorem.

THEOREM 0. *Let M be a positive integer. If Assumptions* 1 *through* 6 *hold, then there exists a partition of consumers into M communities* (*where a community is defined to have a strictly positive population*) *which is an equilibrium.*

To prove this theorem we shall use Brouwer's Fixed Point Theorem. It states that if a continuous function maps each point in a compact convex set to a point in the same compact convex set, then there exists a fixed point within the compact convex set; i.e., if a continuous function f maps each point in a compact convex set X to a point in X, then there exists an $x^* \in X$ such that $x^* \xrightarrow{f} x^*$. The basic strategy behind proving Theorem 0 is to find a compact convex set and a continuous mapping whose fixed point represents an equilibrium. Theorems 1–6 will be used to prove Theorem 0. (The proofs of Theorems 1–6 appear in [9, pp. 36–78].)

THEOREM 1. *If there exists an equilibrium, then each community is formed from a single interval of consumers.*

As a consequence of Theorem 1, we need only consider partitions of [0, 1] into M intervals when searching for an equilibrium with M communities.

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[4] An example of a set of utility functions which satisfy Assumptions 5 and 6 are the Cobb–Douglas functions ${}^{n}u({}^{n}x, {}^{n}z) = {}^{n}x^{\alpha}\,{}^{n}z^{1-\alpha}$, where $\alpha = (1 + 2n)/4$. Clearly,

$$\left(\frac{{}^{n}u_1}{{}^{n}y\,{}^{n}u_2}\right)\bigg|_{(x,t)} = \left(\frac{1+2n}{3-2n}\right)\left(\frac{1-t}{x}\right).$$

We represent such a partition in Fig. 7. The n_j's are the boundaries of the communities; i.e., community i is composed of the consumers between (and perhaps including) consumers n_{i-1} and n_i. The boundaries can be used to describe the community voting equilibria of the respective communities. In order for this to be shown a new concept will be developed. Suppose a community has income $\bar{y}$ ($\bar{y} > 0$). Consider the utility maximizing choice of a consumer who can choose from any point on this community's budget constraint (see Fig. 8). We shall call this point $({}^n\tilde{x}(\bar{y}), {}^n\tilde{t}(\bar{y}))$. From our assumptions concerning consumer endowments and preferences, it is clear that $({}^n\tilde{x}(\bar{y}), {}^n\tilde{t}(\bar{y}))$ is an interior solution; i.e.,

$$(0, 0) \lneqq ({}^n\tilde{x}(\bar{y}), {}^n\tilde{t}(\bar{y})) \lneqq ((\bar{y}/c), 1).$$

This follows from the fact that a consumer prefers a bundle containing positive levels of both goods to a bundle containing none of one good. Due to our strict quasi-concavity assumption, $({}^n\tilde{x}(\bar{y}), {}^n\tilde{t}(\bar{y}))$ is unique. Also, note that

$$\left(\frac{{}^n u_1}{{}^n y'\;{}^n u_2}\right)\Bigg|_{({}^n\tilde{x}(\bar{y}), {}^n\tilde{t}(\bar{y}))} = -\frac{c}{\bar{y}}.$$

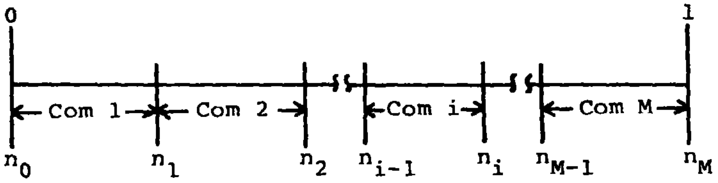

FIGURE 7

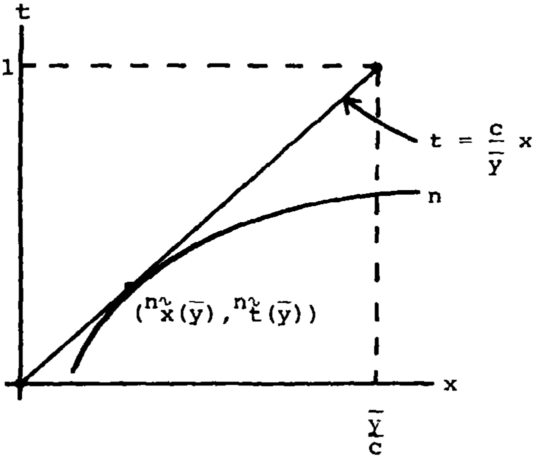

FIGURE 8

We shall now show that such a utility-maximizing choice of a community's median consumer is a community voting equilibrium; i.e., community i is in voting equilibrium if $(x^i, t^i) = ({}^{p^i}\tilde{x}(\bar{y}^i), {}^{p^i}\tilde{t}(\bar{y}^i))$, where

$$p^i = \frac{n_{i-1} + n_i}{2} = \text{community } i\text{'s median consumer,}$$

and

$$\bar{y}^i = \int_{n_{i-1}}^{n_i} {}^{n}y \, dn = \text{community } i\text{'s income.}$$

Recall that for (x^i, t^i) to be a community voting equilibrium of community i,

(1) (x^i, t^i) must lie on community i's budget constraint, $t = (c/\bar{y}^i)x$, to ensure a balanced budget, and

(2) $({}^{n}u_1/{}^{n}y\,{}^{n}u_2)|_{(x^i,t^i)} \geqslant c/\bar{y}^i$ for at least half of community i's residents and $({}^{n}u_1/{}^{n}y\,{}^{n}u_2)|_{(x^i,t^i)} \leqslant c/\bar{y}^i$ for at least half of its residents to ensure majority rule.

From its definition, $({}^{p^i}\tilde{x}(\bar{y}^i), {}^{p^i}\tilde{t}(\bar{y}^i))$ lies on community i's budget constraint and therefore the balanced budget requirement is met. Next, observe that

$$({}^{p^i}u_1/{}^{p^i}y\,{}^{p^i}u_2)|_{({}^{p^i}\tilde{x}(\bar{y}^i),{}^{p^i}\tilde{t}(\bar{y}^i))} = c/\bar{y}^i.$$

$({}^{n}u_1/{}^{n}y\,{}^{n}u_2)\,|_{({}^{p^i}\tilde{x}(\bar{y}),{}^{p^i}\tilde{t}(\bar{y}))}$ is an increasing function of n (Assumption 5); hence,

$$\left(\frac{{}^{n}u_1}{{}^{n}y\,{}^{n}u_2}\right)\Bigg|_{({}^{p^i}\tilde{x}(\bar{y}^i),{}^{p^i}\tilde{t}(\bar{y}^i))} \lesseqgtr \frac{c}{\bar{y}^i} \qquad \text{over all } n \lesseqgtr p^i.$$

Since p^i is community i's median consumer, majority rule prevails. In our search for an equilibrium we can restrict our attention to cases in which each community is in voting equilibrium; hence, we shall set

$$(x^i, t^i) = ({}^{p^i}\tilde{x}(\bar{y}^i), {}^{p^i}\tilde{t}(\bar{y}^i)) \qquad \text{over all } i = 1, 2, \ldots, M.$$

The fact that Theorem 1 permits us to consider only partitions of [0, 1] into M intervals not only allows us to represent community voting equilibria easily but also aids us in finding the compact convex set needed to apply Brouwer's Fixed Point Theorem. Recall that such partitions are represented as shown in Fig. 9. Note that the lower boundary of community 1, n_0, is 0 and that the upper boundary of community M, n_M, is 1. Letting w_i equal the population of community i, it is clear that

$$w_i = n_i - n_{i-1} > 0 \qquad \text{over all} \quad i = 1, 2, \ldots, M.$$

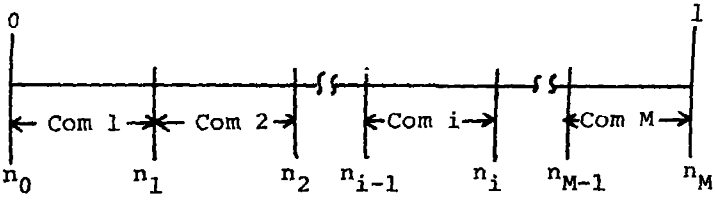

FIGURE 9

A partition of consumers into M communities can be uniquely represented by the vector $\bar{w} = (w_1, w_2, \ldots, w_M)$. $\bar{w}$ is a point on the M-dimensional unit simplex because

$$\begin{aligned}\sum_{j=1}^{M} w_j &= (n_M - n_{M-1}) + (n_{M-1} - n_{M-2}) + \cdots + (n_1 - n_0) \\ &= n_M - n_0 \\ &= 1.\end{aligned}$$

Hence, in our search for an equilibrium we need only consider partitions which are represented by the points on the interior of the M-dimensional unit simplex.

Unfortunately, Brouwer's Fixed Point Theorem cannot be applied directly to the set of points on the interior of the M-dimensional unit simplex because this set is not closed. The entire M-dimensional unit simplex is also unsuitable for it contains points (all those where some $w_j = 0$) that represent partitions in which the population of a community equals zero. To solve this problem we shall show that an equilibrium which requires positive community size, in fact requires that every community exceed some minimal size. Communities which are too "small" are dominated by some "larger" community. Theorem 2 demonstrates this fact.

THEOREM 2. *If there exists an equilibrium with M communities, all communities must have a population greater than a strictly positive constant,* μ_M, *where* $\mu_M < 1/M$.

As a consequence of this result we need only be concerned with partitions of the consumers in which all communities have a population greater than μ_M. The set of all points on the M-dimensional unit simplex such that

$$w_i \geqslant \mu_M \qquad \text{over all } i = 1, 2, \ldots, M$$

contains all $\bar{w}$ which represent partitions of this type. This set will be called the core of the M-dimensional unit simplex.[5] Since the core is a compact convex set it meets the conditions required by Brouwer's Fixed Point Theorem.

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[5] Do not confuse this with the core of an economy.

The next objective is to find a continuous mapping whose fixed point represents an equilibrium with M communities. Let us define a function, $\beta_i^j(n, \bar{w})$, which indicates whether a resident of community i desires to switch to community j; in particular, for each $j \neq i$

$$\beta_i^j(n, \bar{w}) = \begin{cases} 1 & \text{if } (x^j, t^j)\, R_n\, (x^i, t^i), \\ 0 & \text{if } (x^i, t^i)\, P_n\, (x^j, t^j), \end{cases}$$

where the (x^k, t^k)'s are the community voting equilibria associated with the point $\bar{w}$. Furthermore, let

$$\beta_i(n, \bar{w}) = \max_{j \neq i} \beta_i^j(n, \bar{w}).$$

Clearly, $\beta_i(n, \bar{w})$ equals 1 if and only if $(x^j, t^j)\, R_n (x^i, t^i)$ for some $j \neq i$ and 0 otherwise. Since community i's boundaries are n_{i-1} and n_i, the number of consumers in community i who desire to switch is given by

$$\lambda_i(\bar{w}) = \int_{n_{i-1}}^{n_i} \beta_i(n, \bar{w})\, dn.$$

Hence, an equilibrium with M communities is described by a point $\bar{w}$ on the core of the M-dimensional unit simplex where

$$\lambda_i(\bar{w}) = 0 \qquad \text{over all } i = 1, 2, \ldots, M.$$

Note that since $w_i = n_i - n_{i-1}$,

$$0 \leqslant \lambda_i(\bar{w}) \leqslant w_i \qquad \text{over all } i = 1, 2, \ldots, M.$$

Consider a mapping f which is defined over all points on the core of the M-dimensional unit simples $f\colon \bar{w} \xrightarrow{f} \bar{w}'$, where

$$w_i' = \frac{w_i + \lambda_i(\bar{w})}{1 + \sum_{j=1}^{M} \lambda_j(\bar{w})} \qquad \text{over all } i = 1, 2, \ldots, M.$$

THEOREM 3. *f is a continuous mapping from a point on the core of the M-dimensional unit simplex to a point on the M-dimensional unit simplex.*

Unfortunately, while f is continuous, it may map a point on the core of the M-dimensional unit simplex to a point on the M-dimensional unit simplex which is not on the core. To remedy this we shall define a new mapping g so that Brouwer's Fixed Point Theorem may be applied to the mapping fg. Let $\bar{w}' \xrightarrow{g} \bar{w}''$ over all $\bar{w}'$ on the M-dimensional unit simplex such that

(1) if $\bar{w}'$ is on the core of the simplex g maps it onto itself;

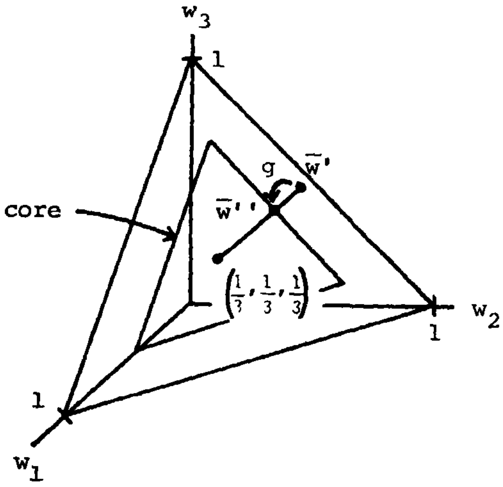

FIGURE 10

(2) if $\bar{w}'$ is not on the core of the unit simplex g maps it to the point which is the intersection of the boundary of the core with the line connecting the points $\bar{w}'$ and $(1/M, 1/M, \ldots, 1/M)$; e.g., for $M = 3$ (see Fig. 10). In particular, g is defined by $\bar{w}' \xrightarrow{g} \bar{w}''$, where

$$w_i'' = w_i' + \left(\frac{1}{M} - w_i'\right)\{\max[h_1(\bar{w}'), h_1(\bar{w}'), \ldots, h_M(\bar{w}')]\}$$

and

$$h_i(\bar{w}') = \begin{cases} \dfrac{\mu_M - w_i'}{(1/M) - w_i'} & \text{if } w_i' < \mu_M, \\ 0 & \text{if } w_i' \geqslant \mu_M, \end{cases}$$

over all $i = 1, 2, \ldots, M$.

$$\begin{tikzcd} \bar{w} \arrow[r, "f"] \arrow[rr, bend right, "fg"'] & \bar{w}' \arrow[r, "g"] & \bar{w}'' \end{tikzcd}$$

Clearly, g is a continuous mapping from a point on the M-dimensional unit simplex to a point on the core of the M-dimensional unit simplex. Hence, fg is a continuous mapping of $\bar{w}$ to a point on the core of the M-dimensional unit simplex.

We can now apply Brouwer's Fixed Point Theorem: There exists a point, $\bar{w}^*$, on the core of the M-dimensional unit simplex which is a fixed point of the mapping fg. Although it was necessary to introduce g to apply

Brouwer's Fixed Point Theorem, Theorem 4 demonstrates that a fixed point of fg is also a fixed point of f:

THEOREM 4. *If* $\bar{w}^*$ *is a fixed point of the mapping* fg, *then* $\bar{w}^*$ *is a fixed point of the mapping* f; *i.e., if* $\bar{w}^* \xrightarrow{fg} \bar{w}^*$, *then* $\bar{w}^* \xrightarrow{f} \bar{w}^*$.

As a consequence of Theorem 4,

$$w_i^* = \frac{w_i^* + \lambda_i(\bar{w}^*)}{1 + \sum_{j=1}^{M} \lambda_j(\bar{w}^*)} \qquad \text{over all } i = 1, 2,\ldots, M;$$

and hence

$$\lambda_i(\bar{w}^*) = w_i^* \sum_{j=1}^{M} \lambda_j(\bar{w}^*) \qquad \text{over all } i = 1, 2,\ldots, M.$$

Recall that $\bar{w}^*$ lies on the core of the M-dimensional unit simplex; hence, all w_i^*'s are greater than zero. Therefore, since $\sum_{j=1}^{M} w_j^* = 1$,

$$\frac{\lambda_1(\bar{w}^*)}{w_1^*} = \frac{\lambda_2(\bar{w}^*)}{w_2^*} = \cdots = \frac{\lambda_k(\bar{w}^*)}{w_k^*} = \cdots = \frac{\lambda_M(\bar{w}^*)}{w_M^*} = \frac{\sum_{j=1}^{M} \lambda_j(\bar{w}^*)}{\sum_{j=1}^{M} w_j^*}.$$

The percentage of consumers in each community who desire to switch, $(\lambda_i(\bar{w}^*)/w_i^*)$, is the same and accordingly is equal to the percentage of consumers in society as a whole who desire to switch, $(\sum_{j=1}^{M} \lambda_j(\bar{w}^*)/\sum_{j=1}^{M} w_j^*)$.

We shall show that $\bar{w}^*$ represents an equilibrium with M communities by arguing that all $\lambda_i(\bar{w}^*)$'s equal zero. To do this two theorems will be considered regarding partitions in which each community is formed from a single interval of consumers.

THEOREM 5. *If each community is formed from a single interval of consumers, then some consumers do not desire to switch; i.e.,*

$$\frac{\sum_{j=1}^{M} \lambda_j(\bar{w})}{\sum_{j=1}^{M} w_j} < 1.$$

Since

$$\frac{\lambda_i(\bar{w}^*)}{w_i^*} = \frac{\sum_{j=1}^{M} \lambda_j(\bar{w}^*)}{\sum_{j=1}^{M} w_j^*} \qquad \text{over all } i = 1, 2,\ldots, M,$$

it follows directly from Theorem 5 that

$$\lambda_i(\bar{w}^*) < w_i^* \qquad \text{over all } i = 1, 2,\ldots, M;$$

i.e., in the partition represented by $\bar{w}^*$, some consumers in each community do not desire to switch. We shall now use this property to argue that there is one community, community k, in which no consumers desire to switch.

THEOREM 6. *If each community is formed from a single interval of consumers and if some consumers in each community do not desire to switch, then there exists (at least) one community in which no consumers desire to switch; i.e., if* $\lambda_i(\bar{w}) < w_i$ *over all* $i = 1, 2, \ldots, M$, *then there exists a* k *such that* $\lambda_k(\bar{w}) = 0$.

Due to Theorem 6, there exists a k such that $\lambda_k(\bar{w}^*) = 0$. Since

$$\frac{\lambda_1(\bar{w}^*)}{w_1^*} = \frac{\lambda_2(\bar{w}^*)}{w_2^*} = \cdots = \frac{\lambda_M(\bar{w}^*)}{w_M^*},$$

it is clear that

$$\lambda_i(\bar{w}^*) = 0 \qquad \text{over all } i = 1, 2, \ldots, M.$$

$\bar{w}^*$ represents an equilibrium.

4. THE RELATIVE PREFERENCE ASSUMPTIONS

Assumptions 5 and 6 place strong restrictions upon relative consumer preferences. We shall now interpret these assumptions and subsequently argue, with the aid of an example, that such restrictions are crucial if the existence of an equilibrium is to be guaranteed. Recall that $({}^n u_1/{}^n y\,{}^n u_2)|_{(x,t)}$ represents consumer n's marginal rate of substitution of the public good for the tax rate when he resides in a community which provides x units of the public good and levies a tax rate of t. Using this insight, an analogy can be drawn between Assumption 5 and single peakedness. The latter assumes that the *alternatives* can be ordered so that for *every consumer*, consumer utility as a function of *alternative rank* has a specified shape, i.e., is single peaked. Assumtion 5 assumes that the *consumers* can be ordered so that for *every alternative*, consumer marginal rate of substitution as a function of *consumer rank* has a specified shape, i.e., is increasing. In other words, if the consumers are ranked in order of their marginal rates of substitution for one particular (x, t), Assumption 5 ensures that this same ranking will rank the consumers in order of their marginal rates of substitution for every other (x, t). (In some sense) Assumption 5 insists that relative consumer preferences are consistent over all (x, t).

Assumption 6 insists that consumer preferences exhibit a complete spectrum of marginal rates of substitution (between those of the "extreme" consumers, consumers 0 and 1). If all such marginal rates of substitution are not represented, an equilibrium typically fails to exist. To argue this point we shall develop a new concept, the "switchover" point, which allows us to illustrate graphically the existence (or nonexistence) of equilibria in the two-community case. Using this new concept it is shown that a two-community equilibrium exists when Assumptions 5 and 6 are satisfied; subsequently, an example is presented to show that when Assumption 6 is not met, an equilibrium need not exist.

Recall that n_1 represents the boundary between the two communities. In Appendix 2 we show that for any two communities there is a point, the "switchover" point, which marks the division between those consumers who prefer one community and those who prefer the other. Informally, let $\hat{n}(n_1)$ equal the switchover point of communities 1 and 2 whenever n_1 is the community boundary[6] (see Fig. 11).

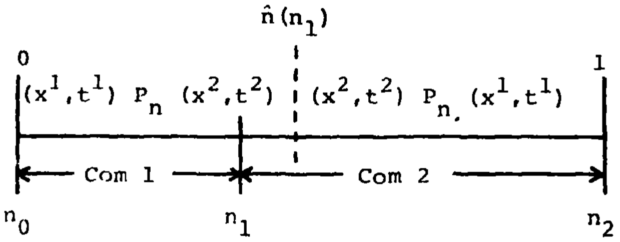

FIGURE 11

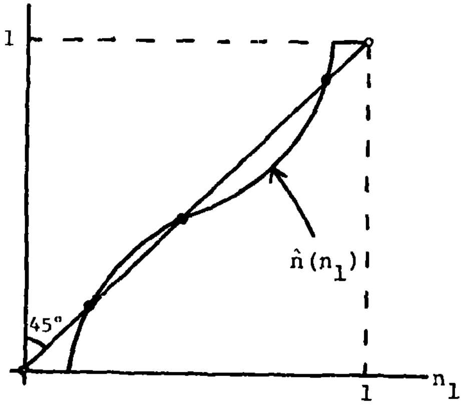

FIGURE 12

A partition is an equilibrium if and only if the switchover point $\hat{n}(n_1)$ equals the community boundary. Clearly, equilibria are characterized by the points at which $\hat{n}(n_1)$ intersects the 45° line (see Fig. 12). To show that $\hat{n}(n_1)$ must intersect this line at least once, we shall argue that $\hat{n}(n_1)$ equals zero (one) for "very low values" ("very high values") of n_1, and that it is continuous.[7] Theorem 2 states that whenever the population of one community was "small" (i.e., not exceeding μ_M) all consumers preferred a "larger" community. Thus, if $n_1 \le \mu_2$ ($\ge (1 - \mu_2)$), all consumers will prefer community 2 (1), and hence the switchover point, $\hat{n}(n_1)$, will equal 0 (1).

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[6] In the following discussion we will implicitly assume that all consumers to the left of $\hat{n}(n_1)$ prefer community 1 and all to the right prefer 2 rather than vice versa. To cover the other possibility a more comprehensive definition is required. See Footnote 7.

To argue that $\hat{n}(n_1)$ is continuous, choose an arbitrary $\bar{n}_1$ and $\epsilon > 0$. Let $(\bar{x}^1, \bar{t}^1)$ and $(\bar{x}^2, \bar{t}^2)$ be the appropriate voting equilibria when $\bar{n}_1$ is the community boundary. Since $\hat{n}(\bar{n}_1)$ is this switchover point,

$$(\bar{x}^1, \bar{t}^1)\ P_{\hat{n}(\bar{n}_1)-\epsilon}\ (\bar{x}^2, \bar{t}^2)$$

and

$$(\bar{x}^2, \bar{t}^2)\ P_{\hat{n}(\bar{n}_1)+\epsilon}\ (\bar{x}^1, \bar{t}^1).$$

Furthermore, since the utility functions are continuous, consumer $\hat{n}(\bar{n}_1) - \epsilon$ has an indifference curve which places $(\bar{x}^1, \bar{t}^1)$ in the better than set and $(\bar{x}^2, \bar{t}^2)$ in the worse than set. Similarly, consumer $\hat{n}(\bar{n}_1) + \epsilon$ has an indifference curve which places $(\bar{x}^2, \bar{t}^2)$ in the better-than set and $(\bar{x}^1, \bar{t}^1)$ in the worse-than set (see Fig. 13). (Note: As drawn in Fig. 13, two indifference curves of two different consumers can intersect only once. For the proof, see Appendix 1.) Around $(\bar{x}^1, \bar{t}^1)$ and $(\bar{x}^2, \bar{t}^2)$ construct the largest possible circles which intersect each indifference curve at most once. Let δ^1 and δ^2 be the radii of the two circles. Clearly, $\delta_1, \delta_2 > 0$.

It is argued in Appendix 3 that the voting equilibria are continuous functions of the community boundaries as a result of Assumptions 5 and 6. Accordingly, there exists a $\delta > 0$ such that over all $|n_1 - \bar{n}_1| < \delta$, community 1's voting equilibrium lies within a δ_1 of $(\bar{x}^1, \bar{t}^1)$, and community 2's

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[7] More formally, over all $0 < n_1 < 1$,

$$\hat{n}(n_1) = w_1 - \lambda_1(\bar{w}) + \lambda_2(\bar{w}) + \frac{\lambda_1(\bar{w})\lambda_2(\bar{w})\{[w_1 - \lambda_1(\bar{w})] - [w_2 - \lambda_2(\bar{w})]\}}{w_1[w_2 - \lambda_2(\bar{w})] + w_2[w_1 - \lambda_1(\bar{w})]},$$

where $\bar{w} = (n_1, (1 - n_1))$. The properties attributed to $\hat{n}(n_1)$ above hold with the formal definition. First, we must show that $\hat{n}(n_1)$ is well defined. This follows directly from Theorem 5; some consumers do not desire to switch; clearly, $w_2 < \lambda_2(\bar{w})$ or $w_1 < \lambda_2(\bar{w})$. Since w_1, $w_2 > 0$, over all $0 < n_1 < 1$, the denominator of the last term is positive and accordingly $\hat{n}(n_1)$ is well defined. To show that a partition is an equilibrium with two communities if and only if $\hat{n}(n_1) = n_1$, we shall argue that if the switchover point is $\hat{n}^{1,2}$, $\hat{n}(n_1) = \hat{n}^{1,2}$ and if the switchover point is $\hat{n}^{2,1}$, $\hat{n}(n_1)$ equals either 0 or 1, where $\hat{n}^{j,k}$ is the switchover point of communities j and k if and only if all consumers to the left of $\hat{n}^{j,k}$ prefer community j and all to the right prefer k. If the switchover point is $\hat{n}^{1,2}$, either $\lambda_1(\bar{w}) = 0$ or $\lambda_2(\bar{w}) = 0$. Thus, $\hat{n}(n_1) = w_1 - \lambda_1(\bar{w}) + \lambda_2(\bar{w})$. Clearly, in this case $\hat{n}(n_1) = \hat{n}^{1,2}$. If the switchover point is $\hat{n}^{2,1}$, either $\lambda_1(\bar{w}) = w_1$ or $\lambda_2(\bar{w}) = w_2$ (but not both, due to Theorem 5); hence, $\hat{n}(n_1)$ equals 0 or 1. Secondly, observe that if $0 < n_1 \le \mu_2$, $\lambda_1(\bar{w}) = w_1$ (Theorem 2); hence, $\hat{n}(n_1) = 0$. Similarly, if $(1 - \mu_2) \le n_1 < 1$, $\hat{n}(n_1) = 1$. Last, since $\lambda_1(\bar{w})$ and $\lambda_2(\bar{w})$ are continuous functions of $\bar{w}$ (see [9, pp. 133–138]) and since $\bar{w}$ is a continuous function of n_1, $\hat{n}(n_1)$ is a continuous function of n_1.

voting equilibrium lies within a δ_2 of $(\bar{x}^2, \bar{t}^2)$. From Fig. 13 it is clear that whenever the two-community voting equilibria lie within there regions, the switchover point will lie within an ϵ of $\hat{n}(\bar{n}_1)$. Thus, when Assumptions 5 and 6 are met, $\hat{n}(n_1)$ is a continuous function of n_1 and $\hat{n}(n_1)$ must intersect the 45° line (at least once), ensuring the existence of an equilibrium.

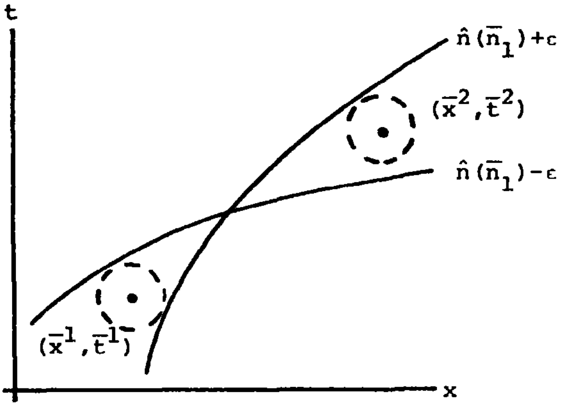

FIGURE 13

Note that the continuity of the community voting equilibria is crucial to the above argument. If the voting equilibria are not continuous functions of the community boundary, the switchover point need not be continuous, $\hat{n}(n_1)$ need not intersect the 45° line, and an equilibrium need not exist. Without Assumption 6, the continuity of the voting equilibria cannot be guaranteed and the existence of an equilibrium cannot be ensured. We shall illustrate this point by considering an example in which consumer preferences do not include all marginal rates of substitution (between those of the extreme consumers). In our example the economy is composed of a continuum of consumers on the interval [0, 1]. (We have not considered an economy with a finite number of discrete consumers because the "discreteness" introduces an additional problem to the analysis: the size of a community cannot be changed by an arbitrarily small amount. To avoid the introduction of this additional restriction, which makes the existence of an equilibrium even more tenuous, we must consider a "continuum economy.") Let each consumer have a Cobb–Douglas utility function

$${}^n u(x, z) = x^{\alpha(n)} z^{(1-\alpha(n))},$$

where

$$\alpha(n) = \begin{cases} 0.2 + 0.5n & \text{over all } 0 \leqslant n \leqslant 0.725, \\ 0.4 + 0.5n & \text{over all } 0.725 < n \leqslant 1. \end{cases}$$

Also, let $^n y = 100$ over all $0 < n \leq 1$, and $c = 1$. Clearly,

$$\left(\frac{^n u_1}{^n y\ ^n u_2}\right)\Bigg|_{(x,t)} \begin{cases} \left(\dfrac{2+5n}{8-5n}\right)\left(\dfrac{1-t}{x}\right) & \text{over all } 0 < n \leq 0.725, \\ \left(\dfrac{4+5n}{6-5n}\right)\left(\dfrac{1-t}{x}\right) & \text{over all } 0.725 < n \leq 1, \end{cases}$$

nd for all $x > 0$, $t < 1$,

$$\lim_{n \to 0.725+} \left(\frac{^n u_1}{^n y\ ^n u_2}\right)\Bigg|_{(x,t)} = 3.2\left(\frac{1-t}{x}\right) \neq 1.3\left(\frac{1-t}{x}\right)$$

$$= \left(\frac{^{0.725}u_1}{^{0.725}y\ ^{0.725}u_2}\right)\Bigg|_{(x,t)}.$$

While Assumption 5 is met, Assumption 6 is not, due to a discontinuity at $n = 0.725$. Accordingly, when community 2's median consumer is 0.725, its oting equilibrium jumps from (31, 0.56) to (42, 0.76) (see Fig. 14).

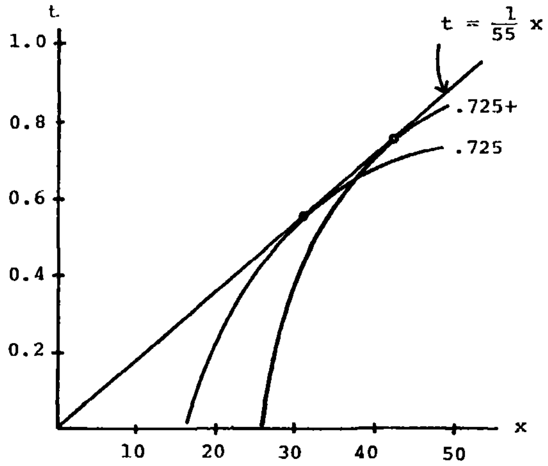

FIGURE 14

Consumer 0.725 is community 2's median whenever consumer 0.45 is the oundary consumer. The discontinuity in community 2's voting equilibrium auses the switchover point to jump from 0.33 to 0.59 when $n_1 = 0.45$, s illustrated in Fig. 15. $\hat{n}(n_1)$ does not intersect the 45° line and accordingly here exists no equilibrium with two communities. When $n_1 \leq 0.45$, the witchover point lies below the boundary consumer and accordingly some esidents of community 1 desire to switch to community 2. Furthermore, hen $n_1 > 0.45$, the switchover point lies above the boundary consumer nd some residents of community 2 desire to switch to community 1.

Accordingly, we have shown that rather strong restrictions must be placed upon relative consumer preferences if the existence of an equilibrium is to be guaranteed.

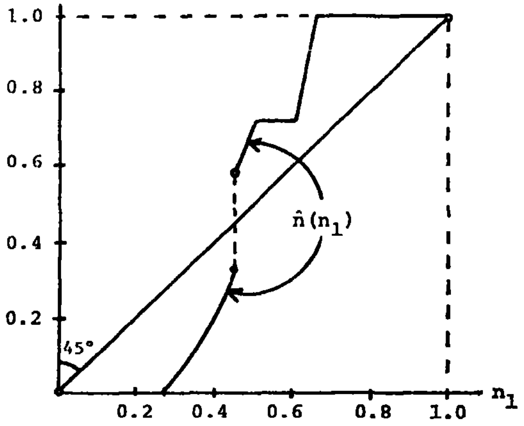

FIGURE 15

5. CONCLUSIONS

We have developed a model in which each consumer chooses his place of residence from a limited number of communities on the basis of the levels of the public good and tax rates offered by the different communities. Each community decides upon its level of the public good and tax rate with an intracommunity majority-rule voting process. The standard restrictions usually placed upon consumer preferences and production technology do not ensure the existence of an equilibrium; only when additional conditions concerning relative consumer preferences are satisfied can existence be guaranteed.

APPENDIX 1

PROPOSITION 1. *Two indifference curves of two different consumers intersect at most once in* (x, t) *space (over all* $x > 0$ *and* $t < 1$).

We shall prove this by contradiction. Consider two consumers, n^* and n^{**}, with indifference curves as shown in Fig. 16. Recall that the slope of consumer n's indifference curve at the point (x, t) is just $({}^{n}u_1/{}^{n}y\,{}^{n}u_2)|_{(x,t)}$. Therefore, at point $(\bar{x}, \bar{t})$,

$$\left(\frac{{}^{n^{**}}u_1}{{}^{n^{**}}y\;{}^{n^{**}}u_2}\right)\bigg|_{(\bar{x},\bar{t})} < \left(\frac{{}^{n^{*}}u_1}{{}^{n^{*}}y\;{}^{n^{*}}u_2}\right)\bigg|_{(\bar{x},\bar{t})}.$$

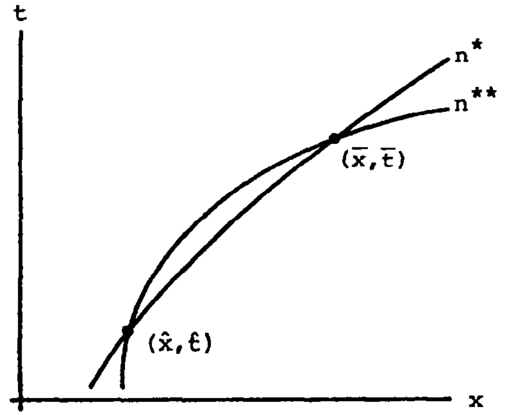

FIGURE 16

Since $({}^{n}u_1/{}^{n}y\,{}^{n}u_2)|_{(\hat{x},\hat{t})}$ is an increasing function of n (Assumption 5), $n^{**} < n^{*}$. But also, observe that at point $(\hat{x}, \hat{t})$,

$$\left(\frac{{}^{n^{**}}u_1}{{}^{n^{**}}y\;{}^{n^{**}}u_2}\right)\Bigg|_{(\hat{x},\hat{t})} > \left(\frac{{}^{n^{*}}u_1}{{}^{n^{*}}y\;{}^{n^{*}}u_2}\right)\Bigg|_{(\hat{x},\hat{t})}.$$

Clearly, this implies that $n^{**} > n^{*}$, a contradiction.

APPENDIX 2

PROPOSITION 2. *If* $(\hat{x}, \hat{t}) \neq (\bar{x}, \bar{t})$ *such that* $\hat{x} \leq x$ *and* $\hat{x} > 0$, $\hat{t} < 1$ *or* $\bar{x} > 0$, $\bar{t} < 1$, *then there exists an* $\tilde{n}$ *such that*

$$(\hat{x}, \hat{t})\; P_n\; (\bar{x}, \bar{t}) \qquad \textit{over all } 0 \leqslant n < \tilde{n}$$

and

$$(\bar{x}, \bar{t})\; P_n\; (\hat{x}, \hat{t}) \qquad \textit{over all } \tilde{n} < n \leqslant 1.$$

We shall prove this by defining two sequences, $\{c_i\}$ and $\{d_i\}$, and arguing (with the aid of three lemmas) that their limits are identical and equal to the desired $\tilde{n}$. Let

$$c_0 = 0,$$

$$d_0 = 1,$$

and for all $i = 1, 2, \ldots$,

$$c_i = \begin{cases} e_{i-1}, & \text{if } (\hat{x}, \hat{t})\; R_{e_{i-1}}\; (\bar{x}, \bar{t}), \\ c_{i-1}, & \text{if } (\bar{x}, \bar{t})\; P_{e_{i-1}}\; (\hat{x}, \hat{t}), \end{cases}$$

and

$$d_i = \begin{cases} e_{i-1}, & \text{if } (\bar{x}, \bar{t})\, P_{e_{i-1}}\, (\hat{x}, \hat{t}), \\ d_{i-1}, & \text{if } (\hat{x}, \hat{t})\, R_{e_{i-1}}\, (\bar{x}, \bar{t}), \end{cases}$$

where $e_{i-1} = (c_{i-1} + d_{i-1})/2$.

Lemma 1. *The sequences* $\{c_i\}$ *and* $\{d_i\}$ *have limits of* c *and* d*, respectively, such that* $c = d$.

In order to show that the sequence $\{c_i\}$ has a limit we shall first argue that $0 \leqslant c_i < d_i \leqslant 1$ by induction; i.e., suppose that $0 \leqslant c_{i-1} < d_{i-1} \leqslant 1$. Since $e_{i-1} = (c_{i-1} + d_{i-1})/2$, it is clear that $0 \leqslant c_{i-1} < e_{i-1} < d_{i-1} \leqslant 1$. Thus, if $(\hat{x}, \hat{t})\, R_{e_{i-1}}(\bar{x}, \bar{t})$,

$$0 \leqslant c_{i-1} < e_{i-1} = c_i < d_i = d_{i-1} \leqslant 1,$$

or if $(\bar{x}, \bar{t})\, P_{e_{i-1}}(\hat{x}, \hat{t})$,

$$0 \leqslant c_{i-1} = c_i < d_i = e_{i-1} < d_{i-1} \leqslant 1.$$

In either case, $0 \leqslant c_i < d_i \leqslant 1$. Thus, we can complete our induction argument by observing that $0 = c_0 < d_0 = 1$.

Next, note that in either case, $c_{i-1} \leqslant c_i < 1$. Thus, $\{c_i\}$ is a nondecreasing sequence whose elements do not exceed 1. Clearly, $\{c_i\}$ has a limit, c, which does not exceed 1; i.e., $\lim\{c_i\} = c \leqslant 1$. Furthermore, since $\{c_i\}$ is nondecreasing, $c_i \leqslant c$ over all i. Similarly, we can show that $\lim\{d_i\} = d \geqslant 0$ and $d_i \geqslant d$ over all i.

To complete the proof of this lemma, we shall show that $c = d$ by contradiction. First, suppose that $d < c$ (see Fig. 17). Let $\epsilon = \frac{1}{2}(c - d)$. Since $\epsilon > 0$, there exists a k' such that $|c - c_i| < \epsilon$ over all $i \geqslant k'$. Similarly, there exists a k'' such that $|d - d_i| < \epsilon$ over all $i \geqslant k''$. Let $k = \max(k', k'')$. Clearly, $|c - c_k| < \epsilon$ and $|d - d_k| < \epsilon$; hence, $d_k < c_k$, a contradiction.

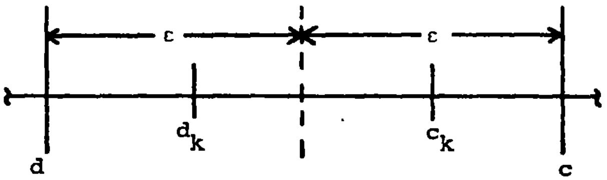

Figure 17

Second, suppose that $c < d$ (see Fig. 18). Let $\epsilon = d - c$. By the same procedure as outlined above choose k such that $|c - c_k| < \epsilon$ and $|d - d_k| < \epsilon$. Now recall that $c_i \leqslant c$ over all i, and $d_i \geqslant d$ over all i.

Hence, $c_k < c$ and $d_k > d$. From Fig. 18 it is clear that $c < (c_k + d_k)/2 < d$; i.e., $c < e_k < d$. From the definition of $\{c_i\}$ and $\{d_i\}$, either $c_{k+1} = e_k$ or $d_{k+1} = e_k$. Hence, either $c_{k+1} > c$ or $d_{k+1} < d$, a contradiction. Thus, the proof of Lemma 1 is complete.

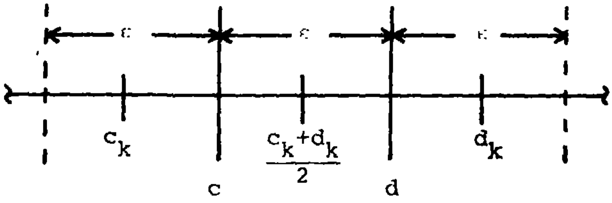

FIGURE 18

Let $\tilde{n} = c = d$. We shall show that

$$(\hat{x}, \hat{t})\, P_n\, (\bar{x}, \bar{t}) \qquad \text{over all } 0 \leqslant n < \tilde{n}$$

by choosing an arbitrary n^*, $0 \leqslant n^* < \tilde{n}$, showing that there exists a $c_k > n^*$ such that $(\hat{x}, \hat{t})\, R_{c_k}\, (\bar{x}, \bar{t})$ (Lemma 2), and using this fact to argue that $(\hat{x}, \hat{t})\, P_{n^*}\, (\bar{x}, \bar{t})$ (Lemma 3).

LEMMA 2. *If* $0 \leqslant n^* < \tilde{n}$, *then there exists a* $c_k > n^*$ *such that* $(\hat{x}, \hat{t})\, R_{c_k}\, (\bar{x}, \bar{t})$.

Let $\epsilon = \tilde{n} - n^*$ (see Fig. 19). Since $\tilde{n} = c$ and $\epsilon > 0$, there exists a k such that

$$| \tilde{n} - c_k | < \epsilon.$$

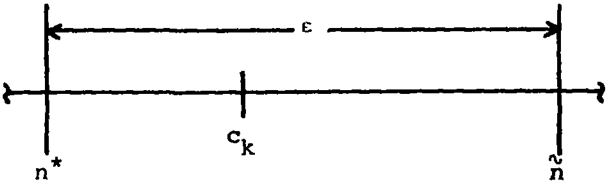

FIGURE 19

Clearly, $c_k > n^*$. We shall show that $(\hat{x}, \hat{t})\, R_{c_k}\, (\bar{x}, \bar{t})$ by contradiction; i.e., suppose that $(\bar{x}, \bar{t})\, P_{c_k}\, (\hat{x}, \hat{t})$. Recall that

$$c_k = \begin{cases} e_{k-1} & \text{if } (\hat{x}, \hat{t})\, R_{e_{k-1}}(\bar{x}, \bar{t}), \\ c_{k-1} & \text{if } (\bar{x}, \bar{t})\, P_{e_{k-1}}\, (\hat{x}, \hat{t}). \end{cases}$$

If $(\hat{x}, \hat{t})\, R_{e_{k-1}}\, (\bar{x}, \bar{t})$, $c_k = e_{k-1}$ and $(\hat{x}, \hat{t})\, R_{c_k}\, (\bar{x}, \bar{t})$. But since $(\bar{x}, \bar{t})\, P_{c_k}\, (\hat{x}, \hat{t})$ by supposition, $(\bar{x}, \bar{t})\, P_{e_{k-1}}\, (\hat{x}, \hat{t})$. Hence, $c_k = c_{k-1}$ and $(\bar{x}, \bar{t})\, P_{c_{k-1}}\, (\hat{x}, \hat{t})$. By repeating this logic we can show that

$$c_k = c_{k-1} = \cdots = c_1 = c_0\,.$$

We have chosen c_k so that $c_k > n^*$. Since $n^* \geqslant 0$ by assumption, $c_k > 0$; hence, $c_0 > 0$, a contradiction.

LEMMA 3. *Suppose that* $(\hat{x}, \hat{t}) \neq (\bar{x}, \bar{t})$ *such that* $\hat{x} \leqslant \bar{x}$ *and* $\hat{x} > 0$, $\hat{t} < 1$ *or* $\bar{x} > 0$, $\bar{t} < 1$. *If* $n^* < c_k$ *and* $(\hat{x}, \hat{t})\, R_{c_k}\, (\bar{x}, \bar{t})$, *then* $(\hat{x}, \hat{t})\, P_{n^*}\, (\bar{x}, \bar{t})$.

To prove this lemma, we shall first argue by contradiction that $\hat{x} > 0$, $\hat{t} < 1$; i.e., suppose that $\hat{x} = 0$ or $\hat{t} = 1$. Clearly, by assumption of the lemma, $\bar{x} > 0$, $\bar{t} < 1$. Due to part (4) of Assumption 4,

$$(\bar{x}, \bar{t})\, P_n\, (\hat{x}, \hat{t}) \qquad \text{over all } n \in [0, 1].$$

However, $(\hat{x}, \hat{t})\, R_{c_k}\, (\bar{x}, \bar{t})$ by assumption, a contradiction. Accordingly, $\hat{x} > 0$, $\hat{t} < 1$.

Since the slope of the indifference curves passing through $(\hat{x}, \hat{t})$ is an increasing function of n (Assumption 5),

$$\left(-\frac{{}^{n^*}u_1}{{}^{n^*}y\,{}^{n^*}u_2}\right)\bigg|_{(\hat{x},\hat{t})} < \left(-\frac{{}^{c_k}u_1}{{}^{c_k}y\,{}^{c_k}u_2}\right)\bigg|_{(\hat{x},\hat{t})}.$$

Also, recall that as a consequence of Assumption 5 these two indifference curves (see Fig. 20) can intersect only at $(\hat{x}, \hat{t})$ (see Appendix 1). Hence,

$$(\hat{x}, \hat{t})\, P_{n^*}\, (\bar{x}, \bar{t}).$$

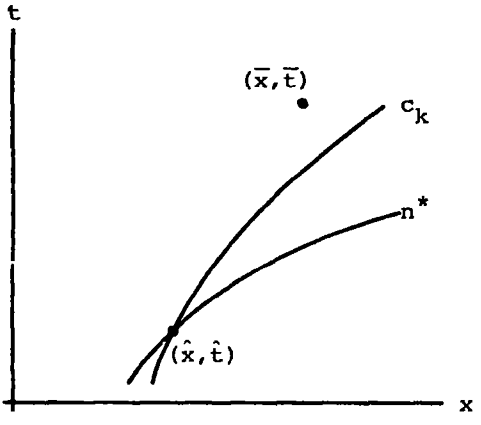

FIGURE 20

Thus, since n^* was chosen arbitrarily such that $0 \leqslant n^* < \tilde{n}$, we have shown that $(\hat{x}, \hat{t})\, P_n\, (\bar{x}, \bar{t})$ over all $0 \leqslant n < \tilde{n}$. Using similar logic we can argue that $(\bar{x}, \bar{t})\, P_n\, (\hat{x}, \hat{t})$ over all $\tilde{n} < n \leqslant 1$. The proof of Proposition 2 is complete.

APPENDIX 3

PROPOSITION 3. *A community voting equilibrium is a (jointly) continuous function of its boundaries; i.e.,* (x^i, t^i) *is a (jointly) continuous function of* n_{i-1} *and* n_i.

Recall that

$$(x^i, t^i) = ({}^{p^i}\tilde{x}(\bar{y}^i), {}^{p^i}\tilde{t}(\bar{y}^i)),$$

where

$$p^i = (n_i + n_{i-1})/2,$$

$$\bar{y}^i = \int_{n_{i-1}}^{n_i} {}^n y \, dn.$$

Clearly, p^i is a continuous function of n_{i-1} and n_i. Since ${}^n y$ is bounded and measurable (Assumption 3), $\bar{y}^i$ is also continuous. Thus, we need only show that $({}^{p^i}\tilde{x}(\bar{y}^i), {}^{p^i}\tilde{t}(\bar{y}^i))$ is a jointly continuous function of p^i and $\bar{y}^i$ over all $0 < p^i < 1$ and $\bar{y}^i > 0$. Choose an arbitrary $\epsilon > 0$ and construct a circle of radius ϵ around $({}^{p^i}\tilde{x}(\bar{y}^i), {}^{p^i}\tilde{t}(\bar{y}^i))$ (Fig. 21).

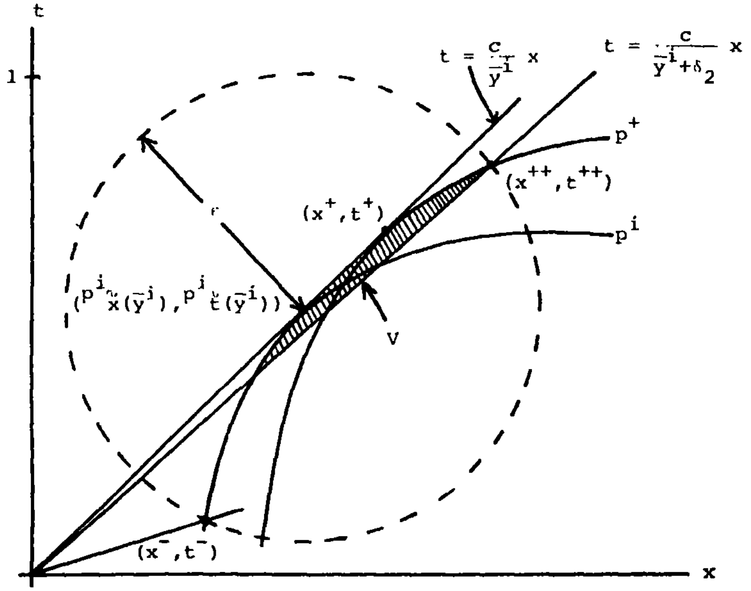
t
1
x
$t = \frac{c}{\bar{y}^i} x$
$t = \frac{c}{\bar{y}^i + \delta_2} x$
p^+
(x^{++}, t^{++})
(x^+, t^+)
p^i
ϵ
$(p^i\tilde{x}(\bar{y}^i), p^i\tilde{t}(\bar{y}^i))$
V
(x^-, t^-)

FIGURE 21

We shall now describe a procedure by which we can obtain a $\delta_1, \delta_2 > 0$ such that if

$$0 \leqslant p - p^i < \delta_1,$$

$$0 \leqslant \bar{y} - \bar{y}^i < \delta_2,$$

then

$$|({}^{p}\tilde{x}(\bar{y}), {}^{p}\tilde{t}(\bar{y})) - ({}^{p^i}\tilde{x}(\bar{y}^i), {}^{p^i}\tilde{t}(\bar{y}^i))| < \epsilon.$$

Let (x^+, t^+) be the point on the line $t = (c/\bar{y}^i)\, x$ midway between $({}^{p^i}\tilde{x}(\bar{y}^i), {}^{p^i}\tilde{t}(\bar{y}^i))$ and the boundary of the circle as illustrated in Fig. 21. Observe that the slope of consumer p^i's indifference curve through (x^+, t^+), $({}^{p^i}u_1/{}^{p^i}y\, {}^{p^i}u_2)|_{(x^+, t^+)}$ is less than $c/\bar{y}^i$, and recall that $({}^{n}u_1/{}^{n}y\, {}^{n}u_2)|_{(x^+, t^+)}$ is a continuous, increasing function of n (Assumptions 5 and 6). If $({}^{1}u_1/{}^{1}y\, {}^{1}u_2)|_{(x^+, t^+)} \geqslant c/\bar{y}^i$, let p^+ be that consumer whose indifference curve has slope of $c/\bar{y}^i$ at (x^+, t^+). If $({}^{1}u_1/{}^{1}y\, {}^{1}u_2)|_{(x^+, t^+)} < c/\bar{y}^i$, redefine (x^+, t^+) to be $({}^{1}\tilde{x}(\bar{y}^i), {}^{1}\tilde{t}(\bar{y}^i))$ and let $p^+ = 1$. In either case, $p^+ > p^i$ and since $({}^{n}\tilde{x}(\bar{y}^i), {}^{n}\tilde{t}(\bar{y}^i))$ is an increasing function of n,

$$(x^+, t^+) = ({}^{p^+}\tilde{x}(\bar{y}^i), {}^{p^+}\tilde{t}(\bar{y}^i)) \gg ({}^{p^i}\tilde{x}(\bar{y}^i), {}^{p^i}\tilde{t}(\bar{y}^i)).$$

Let $\delta_1 = p^+ - p^i$. Clearly, $\delta_1 > 0$.

As illustrated in Fig. 21, define (x^{++}, t^{++}) and (x^-, t^-) as the points at which consumer p^+'s and consumer p^i's indifference curves, respectively, intersect the circle. Consider the two rays from the origin which pass through the points (x^{++}, t^{++}) and (x^-, t^-). Let $t = (c/(\bar{y}^i + \delta_2))\, x$ represent the ray with the higher slope. Clearly,

$$\frac{c}{\bar{y}^i} > \frac{c}{\bar{y}^i + \delta_2},$$

hence, $\delta_2 > 0$.

Now, consider a p and a $\bar{y}$ such that

$$0 \leqslant p - p^i < \delta_1$$

and

$$0 \leqslant \bar{y} - \bar{y}^i < \delta_2 .$$

Clearly,

$$p^i \leqslant p < p^+$$

and

$$\bar{y}^i \leqslant \bar{y} < \bar{y}^i + \delta_2 .$$

$({}^{p}\tilde{x}(\bar{y}), {}^{p}\tilde{t}(\bar{y}))$ lies on the ray $t = (c/\bar{y})\, x$, which itself is within the "wedge" bounded by $t = (c/\bar{y}^i)\, x$ and $t = [c/(\bar{y}^i + \delta_2)]\, x$. Furthermore, since $p^i \leqslant p < p^+$, it is clear that $({}^{p}\tilde{x}(\bar{y}), {}^{p}\tilde{t}(\bar{y}))$ must lie in the shaded region, region V, of that "wedge." Hence, $({}^{p}\tilde{x}(\bar{y}), {}^{p}\tilde{t}(\bar{y}))$ must lie in the circle; i.e.,

$$|({}^{p}\tilde{x}(\bar{y}), {}^{p}\tilde{t}(\bar{y})) - ({}^{p^i}\tilde{x}(\bar{y}^i), {}^{p^i}\tilde{t}(\bar{y}^i))| < \epsilon.$$

Using similar logic we can argue that for an arbitrary $\epsilon > 0$, there exists a $\delta_p, \delta_{\bar{y}} > 0$ such that

$$|({}^{p}\tilde{x}(\bar{y}), {}^{p}\tilde{l}(\bar{y})) - ({}^{p^i}\tilde{x}(\bar{y}^i), {}^{p^i}\tilde{l}(\bar{y}^i))| < \epsilon$$

over all p and $\bar{y}$ such that

$$| p - p^i | < \delta_p$$

and

$$| \bar{y} - \bar{y}^i | < \delta_{\bar{y}} .$$

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Revealed Favorability, Indirect Utility, and Direct Utility*

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This paper studies the axiomatic foundations of the indirect utility function, based upon a revealed preference approach. A chain of comparisons of budgets is regarded as giving a relation on the price-income space (a revealed favorability relation) rather than a relation on the commodity space (a revealed preference relation). The weak and strong axioms of revealed favorability are introduced by analogy with the weak and strong axioms of revealed preference, and the relationship between the former two axioms is investigated. The indirect and direct utility functions are then derived on the basis of the strong axiom of revealed favorability. Neither the continuity property of the demand function nor the convex property of its range is required for the approach taken in the paper.

1. INTRODUCTION AND SUMMARY

The purpose of this paper is to explore the axiomatic foundations of the indirect utility function, based upon a revealed preference approach. There has recently been a strong revival of interest in the indirect utility function, which was first studied by Hotelling [3], Court [1], Roy [14], and others long ago. It is now well known that there exists a basic duality relation between the direct and indirect utility functions: Maximizing the direct utility of commodities is equivalent to minimizing the indirect utility of prices and income, with the identical budget constraint imposed in both instances. In conjunction with this duality relation, a number of propositions on the structure of utility functions and demand systems have been established by Houthakker [5], Samuelson [18, 19], Hicks [2], and others. It is of some interest, therefore, to reexamine the duality relation in the light of revealed preference theory, which has been rather ignored so far.

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We can define the price–income analogs of revealed preference relations as follows. Suppose that the budget set $b(p^0, m^0)$ associated with a price–income vector (p^0, m^0) contains some elements of the choice set $h(p^1, m^1)$ associated with a different price–income vector (p^1, m^1). Then we say that the budget set $b(p^0, m^0)$ is *directly revealed more favorable* than the budget set $b(p^1, m^1)$, or equivalently, that the price-income vector (p^0, m^0) is directly revealed more favorable than the price–income vector (p^1, m^1).[1] This is because there is, presumably, some commodity bundle in $b(p^0, m^0)$ which is better than all commodity bundles in $b(p^1, m^1)$. The weak axiom of revealed favorability asserts that the direct revealed favorability relation thus defined is asymmetric. Needless to say, this is the price-income counterpart of direct revealed preference on the commodity space. Also analogous to the indirect revealed preference relation on the commodity space, the *indirect revealed favorability* relation may be defined as the "transitive closure" of the direct revealed favorability relation on the price–income space. The strong axiom of revealed favorability requires that the indirect revealed favorability relation be asymmetric.

In the traditional revealed preference approach, in order to obtain the representability results, it is always assumed that the range of the demand function has a certain convex property.[2] If the new revealed favorability approach is adopted, however, it will be seen that such a stringent assumption can be entirely eliminated; therefore, the results obtained in this paper help to clarify the role played by the convexity condition on the range of the demand function in the theory of consumer's demand. It should also be noticed that, in contrast to the previous results of Kuga [10] and Weddepohl [22], the continuity or Lipschitzian condition is not imposed on the demand function in this paper.

In Section 2 a set of definitions and assumptions used throughout this paper is introduced. Section 3 discusses the duality relation between revealed favorability relations on the (normalized) price space and revealed preference relations on the commodity space. In Section 4 we establish a theorem concerning the relationship between the weak and strong axioms of revealed favorability; it is seen that the strong axiom holds if and only if both the weak axiom and a certain "regularity condition" hold. The regularity condition is a relationship between two definitions of *income compensation functions* that must hold if the strong axiom holds. Section 5 deals with the representation problem of revealed favorability relations; it is shown that, given the strong axiom of revealed favorability, together with other usual assumptions on the demand function, there exists a real-valued function (an indirect utility function) of price vectors, with the properties of minimality, lower semicontinuity, strict quasi-convexity, and monotonicity. In this final section, a bridge between the indirect and direct utility functions will also be built when the inverse of the demand function exists.

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[1] The term "revealed more favorable" was first used by Weddepohl [22].

[2] For example, see [6, 12, 13, 21].

2. DEFINITIONS AND ASSUMPTIONS

In this section, we shall introduce certain definitions and assumptions used throughout this paper. Revealed favorability relations on the price–income space will be rigorously defined in terms of a system of axioms.

We use the conventional vector notations: For any two vectors $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$, $x \geqq y$ means $x_i \geqq y_i$ for all $i = 1, 2, ..., n$; $x \geq y$ means $x \geqq y$ but $x \neq y$; $x > y$ means $x_i > y_i$ for all $i = 1, 2, ..., n$. Furthermore, $xy = x \cdot y = \sum_i x_i y_i$. Zero vector is denoted by $0 = (0, 0, ..., 0)$.

We are concerned with an individual consumer or a group of consumers who choose a commodity bundle or a set of commodity bundles for given market prices and income. The *commodity space* Y is the set of all conceivable commodity bundles. We assume that it is the nonnegative orthant of the n-dimensional vector space R^n:

$$Y = \{y: y = (y_1, y_2, ..., y_n) \;\&\; 0 \leqq y \in R^n\},$$

where y_i denotes the quantity of commodity i ($i = 1, 2, ..., n$). We do not require that every $y \in Y$ be chosen for some price–income configuration. The *price–income space* $P \times M$ is the positive orthant of the $(n + 1)$-dimensional vector space:

$$P \times M = \{(p, m): (p, m) = (p_1, p_2, ..., p_n, m) \;\&\; 0 < (p, m) \in R^{n+1}\},$$

where p_i denotes the price of commodity i ($i = 1, 2, ..., n$) and m represents the consumer's income. Henceforth, we shall also use another price concept. Let us define the *normalized price space* Q by

$$Q = \{q: q = (q_1, q_2, ..., q_n) \;\&\; q_i = (1/m)\, p_i \quad (i = 1, 2, ..., n) \text{ for some } (p_1, p_2, ..., p_n, m) \in P \times M\}.$$

Q is the set of all conceivable combinations of "relative" prices, expressed in income. The income component can be dropped thereby; it is always equal to unity. Note that the commodity space Y and the (normalized) price space Q have the same dimension, i.e., n.[3]

For a given price vector $q \in Q$, the *budget set* $b(q)$ is defined by

$$b(q) = \{y \in Y: qy \leqq 1\}.$$

---

[3] Note that, in the present case, Q is nothing but the positive orthant of R^n and hence happens to be equal to P.

Let $\mathscr{B}$ be the family of all budget sets:

$$\mathscr{B} = \bigcup \{b(q): q \in Q\}.$$

Let h be a nonempty *demand correspondence* (*function*) on $\mathscr{B}$, that is, a function which to each $b(q)$ assigns a nonempty subset (*choice set*) $h(b(q))$. For $h(b(q))$ and $b(q) \in \mathscr{B}$, we shall also write $h(q)$ and $q \in \mathscr{B}$, respectively. Further, let X be the range of the demand correspondence h:

$$\begin{aligned} X = h(\mathscr{B}) &= \bigcup \{h(q): q \in \mathscr{B}\} \\ &= \{x \in Y: x \in h(q) \text{ for some } q \in Q\}. \end{aligned}$$

Note that X need not be equal to Y, the commodity space; the former set may be a proper subset of the latter. It is assumed that

(H) $qx = 1$, for all $q \in Q$, $x \in h(q)$.

Assumption (H) requires that the whole budget be spent.

We define revealed favorability relations on Q in terms of h and b as follows. Suppose that the budget set $b(q^0)$ associated with a price vector q^0 contains some elements of the choice set $h(q^1)$ associated with a distinct price vector q^1. Then we say that $b(q^0)$ is *directly revealed more favorable* than $b(q^1)$, and we write $b(q^0)\,F^1 b(q^1)$, since there is presumably some commodity bundle in $b(q^0)$ that is better than all commodity bundles in $h(q^1)$. We formally write this as

$$\begin{aligned} b(q^0)\,F^1 b(q^1) &\Leftrightarrow b(q^0) \cap h(q^1) \neq \emptyset \text{ and } q^0 \neq q^1, \\ &\Leftrightarrow q^0x^1 \leqq 1 \text{ for some } x^1 \in h(q^1) \text{ and } q^0 \neq q^1, \end{aligned}$$

where $A \Leftrightarrow B$ indicates that two statements A and B are equivalent. For $b(q^0)\,F^1 b(q^1)$, we shall simply write $q^0F^1q^1$. If $q^0F^1r^1F^1 \cdots F^1r^kF^1q^1$ for a finite sequence $r^1, \ldots, r^k \in Q$, then we say that q^0 is *revealed more favorable in k steps* than q^1, and we write $q^0F^kq^1$. If $q^0F^kq^1$ for some finite integer k, then we say that q^0 is *indirectly revealed more favorable* than q^1, and we write q^0Fq^1; therefore, F is the "transitive closure" of F^1, i.e., the smallest transitive relation including F^1 on Q. It is easy to see that $q^0 \leqq q^1$ implies $q^0F^1q^1$, and hence q^0Fq^1.

Corresponding to these two relations F^1 and F on Q, we introduce two axioms of revealed favorability, the *weak axiom* (WF) and the *strong axiom* (SF):

(WF) for $q^0, q^1 \in Q$, $q^0F^1q^1$ implies $q^1\bar{F}^1q^0$;

(SF) for $q^0, q^1 \in Q$, q^0Fq^1 implies $q^1\bar{F}q^0$;

where $\bar{F}^1$ and $\bar{F}$ denote the negation of F^1 and the negation of F, respectively. These axioms assert that the consumer's behavior should be "directly or indirectly" consistent; therefore, they are quite analogous to the traditional revealed preference axioms of Samuelson [17] and Houthakker [4] on Y, the commodity space.

3. Revealed Favorability and Revealed Preference

In this section we shall briefly discuss the duality relation between revealed favorability relations on the price space and revealed preference relations on the commodity space. We shall also be interested in the related question of how this duality relation corresponds to the more familiar one between the indirect and direct utility functions.

Revealed preference relations on Y are defined in terms of h and b as follows. If there is $q \in Q$ such that $x \in h(q)$, $y \in b(q)$, and $x \neq y$, then we say that x is *directly revealed preferred* to y, and we write xSy. If there is some finite sequence $u^1, \ldots, u^k$ of elements of Y such that $xSu^1S \cdots Su^kSy$, we say that x is *indirectly revealed preferred* to y, and we write xHy. The *weak axiom* (WP) *of revealed preference* asserts that S is asymmetric:

(WP) For $x, y \in Y$, xSy implies $y\bar{S}x$.

On the other hand, the *strong axiom* (SP) *of revealed preference* asserts that H is asymmetric:

(SP) For $x, y \in Y$, xHy implies $y\bar{H}x$.

It is easy to show that if h satisfies (WF) on Q, then for any $x \in X$, the inverse image of x by h has only a single element; therefore, h is uniquely invertible. In contrast, as shown by Samuelson [17, pp. 111 112], if h satisfies (WP) on X, then for any $q \in Q$, the image of q by h is a one-element set; therefore, h is single valued. Hence, (WF) does not imply, and is not implied by, (WP). Likewise, (SF) does not imply, and is not implied by, (SP). To make this point more sharply, consider simple examples of indifference curves on Q with "pointed" portions, or equivalently, those of indifference curves on X with "flat" portions. They indicate the consumer satisfying (WF), but not (WP) (cf. Fig. 1). On the other hand, as is clear in Fig. 2, symmetrical examples of indifference curves on Q with "flat" portions (and hence those of indifference curves on X with "pointed" portions) indicate the consumer satisfying (WP), but not (WF). If h is a one-to-one correspondence between Q and X, however, then it is readily seen that (WF) holds on Q if and only if (WP) holds on X, and also that (SF) holds on Q if and only if (SP) holds on X.[4]

---

[4] We may also develop the price analog of the *congruence axiom of revealed preference* (cf. [12, 13]). To this end, define relations V and W on the price space as

$$q^0 V q^1 \Leftrightarrow b(q^0) \cap h(q^1) \neq \emptyset;$$
$$q^0 W q^1 \Leftrightarrow q^0 V r^1 V \ldots V r^k V q^1 \quad \text{for some } r^1, \ldots, r^k.$$

The *congruence axiom* (CF) *of revealed favorability* is then defined as:

(CF) If $q^0, q^1 \in Q$, $x \in b(q^0) \cap h(q^1)$, and $q^1 W q^0$, then $x \in h(q^0)$.

The new axiom (CF) can be viewed as an extension of the strong axiom (SF) of revealed favorability to the case in which the inverse image of x by h, $x \in X$, is a subset in Q. This extension problem clearly deserves a separate paper and is not discussed further here.

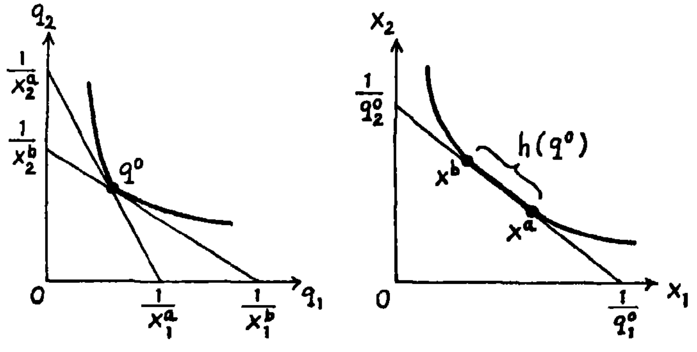

FIG. 1. h satisfies (WF), but not (WP).

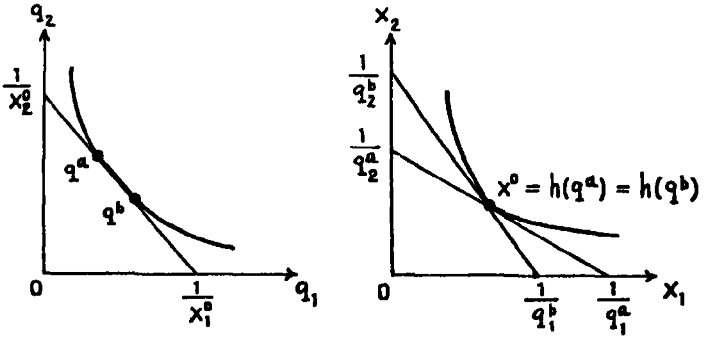

FIG. 2. h satisfies (WP), but not (WF).

4. RELATIONSHIP BETWEEN THE WEAK AND STRONG AXIOMS OF REVEALED FAVORABILITY

In this section we shall study the relationship between the weak and strong axioms of revealed favorability. Suppose that the demand function h satisfies the assumption (H). Then it will be shown that the strong axiom (SF) of revealed favorability holds if and only if the weak axiom (WF) of revealed favorability and a certain regularity condition both hold. The regularity condition is a relationship between two definitions of income compensation functions that must hold if the strong axiom holds.

To this end, we first establish the following lemma.

LEMMA 1. *If the demand function h satisfies* (H) *and* (WF), *then for any* $q^0 \in Q$, *the set* $\{q \in Q\colon q\bar{F}q^0\}$ *is closed in* Q.

Proof.[5] It suffices to show that the set $\{q \in Q\colon qFq^0\}$ is open in Q. Suppose q^1Fq^0. Then, by the definition of F, there exists a price vector $q^2 \in Q$ such that (Fig. 3)

$$1 \geqq q^1x^2 \qquad \text{for some} \quad x^2 \in h(q^2), \quad q^1 \neq q^2, \tag{1}$$

and

$$q^2Fq^0 \qquad \text{or} \qquad q^2 = q^0. \tag{2}$$

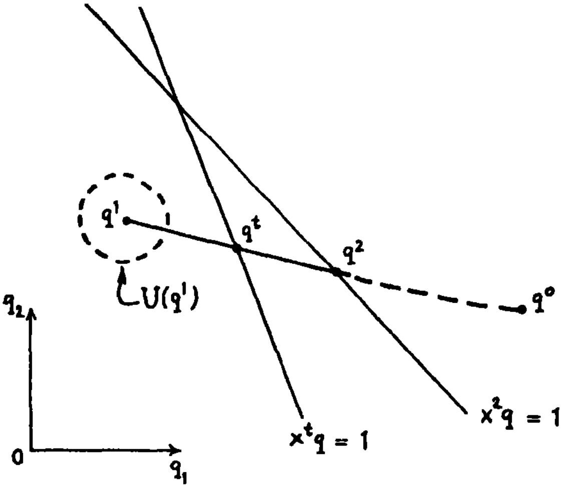

FIGURE 3

Let $q^t = (1-t)\,q^1 + tq^2$, $t \in (0, 1)$, and let $x^t \in h(q^t)$. Then $q^tx^2 = (1-t)\,q^1x^2 + t$. This, together with (1), yields

$$1 \geqq q^tx^2, \qquad q^t \neq q^2. \tag{3}$$

It follows from (3) and (WF) that $q^2x^t > 1$. But, by the choice of q^t, we find

$$(1-t)(q^1x^t - 1) + t(q^2x^t - 1) = 0.$$

Hence, we obtain $q^1x^t < 1$. Choose a neighborhood $U(q^1)$ of q^1 so that

$$qx^t < 1 \qquad \text{for any} \quad q \in U(q^1). \tag{4}$$

Relations (2), (3), and (4) yield

$$qFq^0 \qquad \text{for any} \quad q \in U(q^1);$$

which ensures that the set $\{q \in Q\colon qFq^0\}$ is open in Q. Q.E.D.

---

[5] The method of proof presented here is very close to that used by Uzawa in the proof of property P.V. [21, pp. 138–139].

For a given $p \in P$ and a given $q^0 \in Q$, let us define the sets of incomes

$$\bar{M}(p, q^0) = \{m \in M: (1/m)\,p\bar{F}q^0\};$$
$$\underline{M}(p, q^0) = \{m \in M: q^0\bar{F}(1/m)p\}.$$

Then the income compensation functions $\bar{m}(p, q^0)$ and $\underline{m}(p, q^0)$ are defined in terms of $\bar{M}(p, q^0)$ and $\underline{M}(p, q^0)$, respectively:

$$\bar{m}(p, q^0) = \sup\{m: m \in \bar{M}(p, q^0)\};$$
$$\underline{m}(p, q^0) = \inf\{m: m \in \underline{M}(p, q^0)\}.$$[6]

For a given "absolute" price vector p and a given "relative" price vector q^0, the function $\bar{m}(p, q^0)$ stands for the supremum of incomes m such that $(1/m)p$ is not indirectly revealed more favorable than q^0; and the function $\underline{m}(p, q^0)$ represents the infimum of incomes m such that q^0 is not indirectly revealed more favorable than $(1/m)p$. It is clear by definitoin that $\bar{M}(p, q^0) \supset (0, \bar{m}(p, q^0))$ and $\underline{M}(p, q^0) \supset (\underline{m}(p, q^0), +\infty)$.

LEMMA 2. *If the demand function h satisfies* (H) *and* (WF), *then for any $(p, q^0) \in P \times Q$, the set $\bar{M}(p, q^0)$ is closed in M.*

Proof. By definition, we have $\bar{M}(p, q^0) = M - \{m \in M: (1/m)\,pFq^0\}$. In order to prove that $\bar{M}(p, q^0)$ is closed in M, it suffices to show that the set $\{m \in M: (1/m)\,pFq^0\}$ is open in M.

Let $(1/m)\,pFq^0$. It follows from Lemma 1 that there exists a neighborhood $U((1/m)p)$ of $(1/m)p \in Q$ so that q^1Fq^0 for any $q^1 \in U((1/m)p)$. Choose a sufficiently small $e > 0$ such that $(1/(m+e))p$, $(1/(m-e))p \in U((1/m)p)$. Then for any $m' \in (m-e, m+e)$, we obtain $(1/m')\,pFq^0$. This indicates that the set $\{m \in M: (1/m)\,pFq^0\}$ is open in M. Q.E.D.

It is noted that according to Lemma 2, we have $\bar{M}(p, q^0) = (0, \bar{m}(p, q^0)]$ if h satisfies (H) and (WF).

LEMMA 3. *If the demand function h satisfies* (H) *and* (SF), *then for any $(p, q^0) \in P \times Q$, the income compensation functions $\bar{m}(p, q^0)$ and $\underline{m}(p, q^0)$ exist and are finite.*

Proof. In order to show that $\bar{m}(p, q^0)$ exists and is finite, it suffices to see that $\bar{M}(p, q^0)$ is nonempty and bounded above.

Let $(p, q^0) \in P \times Q$. Choose a sufficiently small $m^1 \in M$ such that $(1/m^1)p > q^0$. Letting $x^1 \in h((1/m^1)p)$, we find

$$1 = (1/m^1)p \cdot x^1 > q^0x^1,$$

---

[6] Income compensation functions were first introduced by McKenzie [11] and Yokoyama [23], independently, in terms of preference orderings on the commodity space.

so that $q^0F^1(1/m^1)p$ or $q^0F(1/m^1)p$. By virtue of (SF), this gives $(1/m^1)\,p\bar{F}q^0$, i.e., $m^1 \in \overline{M}(p, q^0)$. Therefore, $\overline{M}(p, q^0)$ is nonempty.

Now choose a sufficiently large $m^2 \in M$ such that $(1/m^2)p < q^0$. Letting $x^0 \in h(q^0)$, we have

$$1 = q^0x^0 > (1/m^2)p \cdot x^0,$$

whence $(1/m^2)\,pFq^0$. This shows that m^2 is an upper bound to $\overline{M}(p, q^0)$.

The proof that $\underline{m}(p, q^0)$ exists and is finite proceeds in a similar way. Q.E.D.

We formulate a *regularity condition* in terms of $\overline{m}(p, q^0)$ and $\underline{m}(p, q^0)$:

(R) For any $(p, q^0) \in P \times Q$, $\overline{m}(p, q^0) \geqq \underline{m}(p, q^0)$.

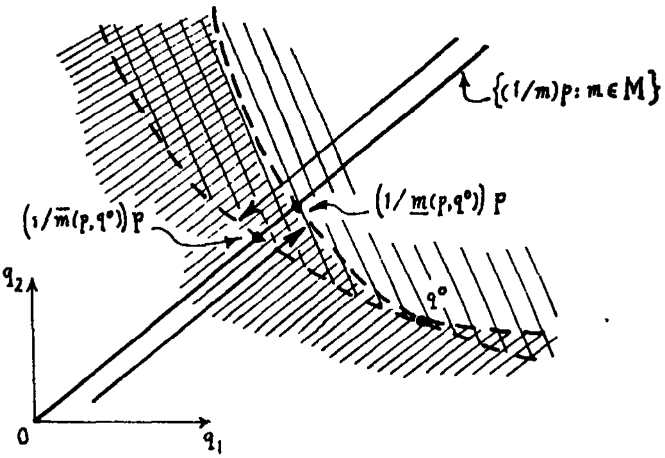

FIG. 4. $\bar{m}(p, q^0) \geqq \underline{m}(p, q^0)$. Note that the upper shaded area denotes the set $\{q \in Q: q\bar{F}q^0\}$ and the lower one denotes the set $\{q \in Q: q^0Fq\}$.

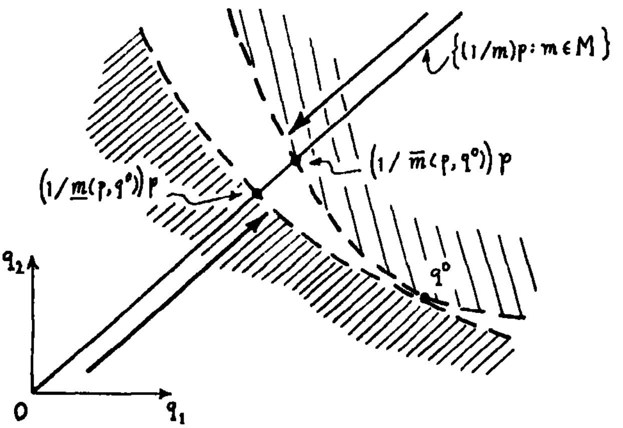

FIG. 5. $\bar{m}(p, q^0) < \underline{m}(p, q^0)$. Note that the upper shaded area denotes the set $\{q \in Q: q\bar{F}q^0\}$ and the lower one denotes the set $\{q \in Q: q^0Fq\}$.

Condition (R) states that for any given price vector (p, q^0), the supremum of incomes m such that $(1/m)p$ is "no better than" q^0 is greater than or equal to the infimum of incomes m such that q^0 is "no better than" $(1/m)p$. Figure 4 represents the case in which (R) is satisfied, while Fig. 5 represents the case in which it is not. As is clear from comparison of these two figures, condition (R) requires that there be no "$\bar{F}$-gap" in the ray $\{(1/m)p: m \in M\}$ for any given price vector p.

We are now in a position to establish a theorem concerning the relation between the weak and strong axioms of revealed favorability.

THEOREM 4. *Suppose that the demand function h satisfies* (H). *Then the strong axiom* (SF) *of revealed favorability holds if and only if both the weak axiom* (WF) *of revealed favorability and the regularity condition* (R) *hold.*

Proof.[7] (a) (Necessity.) Suppose (SF) holds. Then (WF) is obviously implied. Note that $\bar{m}$ and $\underline{\underline{m}}$ exist and are finite by virtue of Lemma 3. Now assume by way of contradiction that (R) does not hold, so $\bar{m}(p, q^0) < \underline{\underline{m}}(p, q^0)$ for some $(p, q^0) \in P \times Q$. Choose an $m \in M$ so that

$$\bar{m}(p, q^0) < m < \underline{\underline{m}}(p, q^0). \tag{5}$$

By the definitions of $\bar{m}$ and $\underline{\underline{m}}$, (5) implies $(1/m)\,pFq^0$ and $q^0F(1/m)p$; which contradicts (SF).

(b) (Sufficiency.) Suppose (WF) and (R) both hold. Let $q^1, q^0 \in Q$ be such that q^1Fq^0. Then we shall show that

$$\bar{m}(q^1, q^0) < 1. \tag{6}$$

To this end, assume otherwise. Since (WF) yields $\bar{M}(q^1, q^0) = (0, \bar{m}(q^1, q^0)]$ by Lemma 2, it would follow that $1 \in \bar{M}(q^1, q^0)$, i.e., $(1/1)\,q^1\bar{F}q^0$, which contradicts our assumption. But, in view of (R), (6) implies that $\underline{\underline{m}}(q^1, q^0) < 1$. Since $(\underline{\underline{m}}(q^1, q^0), +\infty) \subset \underline{\underline{M}}(q^1, q^0)$, we have $1 \in \underline{\underline{M}}(q^1, q^0)$, i.e., $q^0\bar{F}(1/1)\,q^1$. We have thus seen that under (WF) and (R), q^1Fq^0 implies $q^0\bar{F}q^1$. The proof is now complete. Q.E.D.

So far as I know, the relation between the weak and strong axioms of revealed favorability has never been seriously investigated.[8] The hypothesis of Theorem 4 is not so strong, because (i) no continuity condition is imposed on h, and (ii) X, the range of h, need not be convex in Y, the commodity space.

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[7] I am indebted to the referee for suggesting a way to simplify the proof of Theorem 4.

[8] By contrast, the relationship between the weak and strong axioms of revealed preference on the commodity space has been intensively studied so far (cf. [16, 21]).

Remark 1. It should be noticed that the above conditions (i) and (ii) are not implied by (H) and (SF). This point can be demonstrated by two examples, illustrated in Figs. 6 and 7.[9]

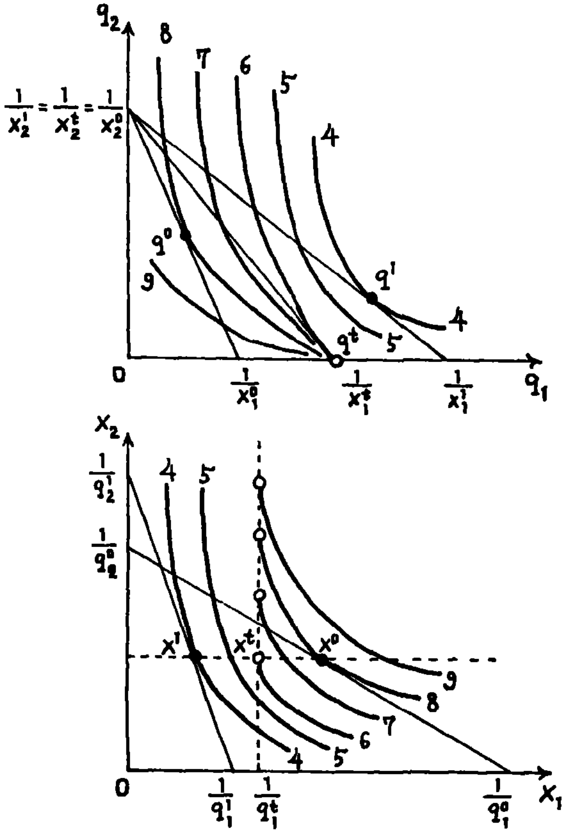

FIG. 6. Let h be generated by indirect preferences indicated in the upper diagram, or equivalently, by direct preferences indicated in the lower diagram. h satisfies (SF) and is continuous. However, the range of h is not convex; indeed, $x^t = (1 - t)x^0 + tx^1 \notin X$.

5. DERIVATION OF THE INDIRECT AND DIRECT UTILITY FUNCTIONS

The duality relation between the direct and indirect utility functions has recently been studied a great deal, especially in connection with special functional forms of separability.[10] In this section, we shall investigate, first, whether the indirect revealed favorability relation on the (normalized) price space can be represented by a real-valued function (i.e., the indirect utility function); and second, whether there is a way of making a bridge between the indirect and direct utility functions from the perspective of the revealed favorability approach taken in this paper.

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[9] Figure 7 is drawn on the basis of Sonnenschein's Example 1 [20, p. 273].

[10] For the duality results, see [7].

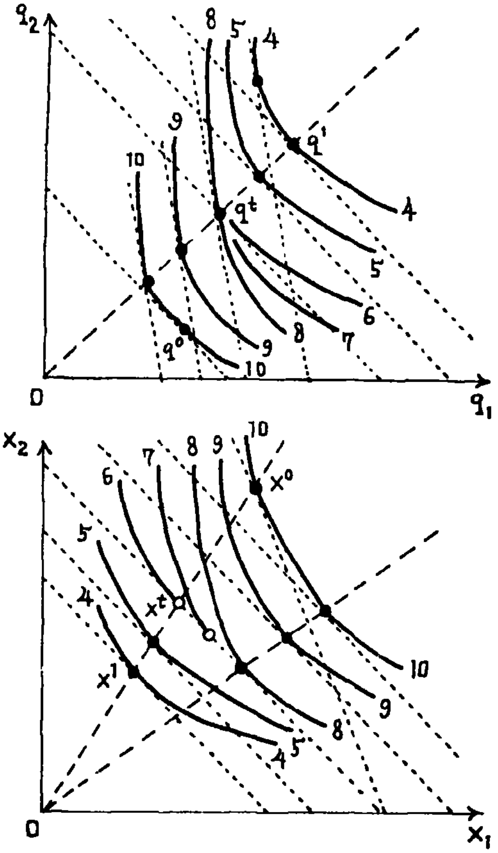

FIG. 7. Let h be generated by indirect preferences indicated in the upper diagram, or equivalently, by direct preferences indicated in the lower diagram. h satisfies (SF), but it is not continuous and its range is not convex. In fact, $x^t = (1 - t)x^0 + tx^1 \notin X$.

We shall state and prove the following interesting result.

THEOREM 5. *Suppose that the demand function h satisfies* (H) *and* (SF). *Then there exists a real-valued function v on Q such that*

(i) *for any $q \in Q$,*

$$h(q) = \{x \in X\colon qx \leqq 1 \ \& \ \forall r_{rx \leqq 1 \,\&\, r \in Q}\ v(r) \geqq v(q)\};$$

(ii) *for any $q \in Q$, the set $\{r \in Q\colon v(q) \geqq v(r)\}$ is closed in Q;*

(iii) *if $v(q^1) \geqq v(q^0)$, $q^1, q^0 \in Q$, $q^1 \neq q^0$, $t \in (0, 1)$, and $q^t = (1 - t)q^1 + tq^0$, then $v(q^1) > v(q^t)$;*

(iv) *if* $q^1 \leq q^0$ *and* $q^1, q^0 \in Q$ *then* $v(q^1) > v(q^0)$;

(v) *for any* $q^1, q^0 \in Q$, q^1Fq^0 *implies* $v(q^1) > v(q^0)$.

Proof.[11] (a) Since F is an asymmetric and transitive relation on Q by (SF) and definition, it is an irreflexive and transitive relation on Q. Let C denote the set of all elements of Q having all coordinates rational. Then we shall show that C is a countable partially F-dense subset of Q. To this end, let $q^1, q^0 \in Q - C$ satisfy q^1Fq^0. By virtue of Lemma 1, we can choose a neighborhood $U(q^1)$ of q^1 so that sFq^0 for all $s \in U(q^0)$. Take a $c \in U(q^1) \cap C$, for which $c \neq q^1$. We then have q^1FcFq^0; therefore C is F-dense in Q. Since F is asymmetric, this clearly implies that C is partially F-dense in Q. Consequently, by [13, Theorem 11, p. 49], there exists a non-negative-valued and bounded function f, $f\colon Q \to R_+$, satisfying

$$\forall q^1, q^0{}_{q^1, q^0 \in Q}\ q^1Fq^0 \Rightarrow f(q^1) > f(q^0). \tag{7}$$

(b) Since the function f is bounded, we can define a real-valued function v on Q by $v = \liminf f$; i.e., for $q \in Q$

$$v(q) = \sup\{\inf\{f(s)\colon s \in U\}\colon q \in U \in \mathfrak{U}\},$$

where $\mathfrak{U}$ denotes the family of open sets in Q. Because v is the lower limit function of f, it is lower semicontinuous (e.g., see [8, Theorem 3.6, p. 75]; which establishes property (ii).

(c) To see property (v), suppose that $q^1, q^0 \in Q$ are such that q^1Fq^0. Since F is lower semicontinuous by Lemma 1, we can choose $q^2 \in Q$ for which $q^1Fq^2Fq^0$, as was shown in (a). By the lower semicontinuity of F again, q^1Fq^2 implies that there exists a neighborhood $U(q^1)$ of q^1 such that qFq^2 for all $q \in U(q^1)$. Consequently we have from (7) that

$$\inf\{f(q)\colon q \in U(q^1)\} \geq f(q^2) > f(q^0).$$

It then follows that

$$\begin{aligned} v(q^1) &= \sup\{\inf\{f(s)\colon s \in U\}\colon q^1 \in U \in \mathfrak{U}\} \\ &\geq \inf\{f(q)\colon q \in U(q^1)\} \geq f(q^2) > f(q^0) \geq v(q^0). \end{aligned}$$

This ensures property (v).

(d) Let

$$w(q) = \{x \in X\colon qx \leq 1 \ \&\ \forall r_{rx \leq 1 \,\&\, r \in Q}\ v(r) \geq v(q)\}.$$

To prove property (i), it suffices to show that for any $q \in Q$, $h(q) = w(q)$. Clearly, from (c), $h(q) \subset w(q)$ for all $q \in Q$. Now suppose $x \in w(q) - h(q)$.

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[11] The method of proof presented here is based on that developed by Richter [12, 13] and Hurwicz and Richter [6].

Since $x \in X$, we have $x \in h(r)$ for some $r \in Q$, $r \neq q$. But we then obtain qFr, which implies $v(q) > v(r)$ by (c). On the other hand, we have $rx = 1$, so that if $x \in w(q)$ we must have $v(r) \geqq v(q)$; which gives a contradiction. Therefore, $h(q) = w(q)$ for all $q \in Q$.

(e) To see property (iii), let $q^1, q^0 \in Q$ be such that $v(q^1) \geqq v(q^0)$ and $q^1 \neq q^0$, and let $q^t = (1-t)\,q^1 + tq^0$ for any $t \in (0, 1)$. Then letting $x^t \in h(q^t)$, we find $(1-t)\,q^1x^t + tq^0x^t = 1$, whence $q^1x^t \leqq 1$ or $q^0x^t \leqq 1$. If $q^1x^t \leqq 1$ then q^1Fq^t, so that $v(q^1) > v(q^t)$ by (c). On the other hand, if $q^0x^t \leqq 1$, then q^0Fq^t, so that $v(q^1) \geqq v(q^0) > v(q^t)$ by assumption and (c). In any case, we thus obtain $v(q^1) > v(q^t)$.

(f) Finally, to prove property (iv), let $q^1, q^0 \in Q$ be such that $q^1 \leq q^0$. Then clearly, q^1Fq^0. In the light of (c), this gives $v(q^1) > v(q^0)$. The proof is now complete. Q.E.D.

Property (i) indicates that for a given price vector q, the choice set $h(q)$ constitutes those commodity bundles x that minimize the indirect utility of price vectors r subject to the budget constraint $rx \leqq 1$. Property (ii) means that the "inferior" set is closed in Q, i.e., that the indirect utility function v is lower semicontinuous. Property (iii) shows that v is strictly quasi-convex, while property (iv) says that it is decreasing. It follows from property (v) that the indirect revealed favorability relation F is represented by v.

In order to make a bridge between the indirect and direct utility functions, it is necessary to introduce an additional restriction on the demand function h:

(D) For any $q \in Q$, $h(q)$ is a singleton.

Recall that the strong axiom (SF) of revealed favorability implies the unique invertibility of h. Therefore, if h satisfies both (SF) and (D), h is a one-to-one correspondence between Q and X. Let us define a function g on X by

$$g(x) = h^{-1}(x), \qquad x \in X.$$

Then it is clear that $g(h(q)) = q$ for all $q \in Q$, and that $h(g(x)) = x$ for all $x \in X$.[12] Since g is the inverse of h on X, it immediately follows from Theorem 5(i) that for any $x \in X$,

$$g(x) = \{q \in Q\colon qx \leqq 1 \;\&\; \forall r_{rx \leqq 1 \,\&\, r \in Q}\, v(r) \geqq v(q)\}.$$

Now we are in a position to define a real-valued function (a direct utility function) u on X by

$$u(x) = v(g(x)), \qquad x \in X.$$

It will be seen in the following theorem that the function u generates the demand function h, i.e., that h uniquely maximizes u over $\mathscr{B}$.

---

[12] For $h(q) = \{x\}$ and $g(x) = \{q\}$, we simply write $h(q) = x$ and $g(x) = q$, respectively.

THEOREM 6. *Suppose that the demand function h satisfies* (H), (SF), *and* (D). *Then there exists a real-valued function u on X such that*

(i) *for any $q \in Q$,*

$$h(q) = \{x : x \in b(q) \cap X \,\&\, \forall y_{y \in b(q) \cap X}\; u(x) \geqq u(y)\},$$

and this set contains only a single element;

(ii) *if $u(x^1) \geqq u(x^0)$, $x^1, x^0 \in X$, $x^1 \neq x^0$, $t \in (0, 1)$, and $x^t = (1 - t)\, x^1 + tx^0 \in X$, then $u(x^t) > u(x^0)$*;

(iii) *if $x^1 \geq x^0$ and $x^1, x^0 \in X$, then $u(x^1) > u(x^0)$*;

(iv) *for any $x^1, x^0 \in X$, $x^1 H x^0$ implies $u(x^1) > u(x^0)$.*

Proof. (a) Under (SF) and (D), h is a one-to-one correspondence between Q and X. Letting $q, r \in Q$, $x = h(q)$, and $y = h(r)$, this implies that xHy if and only if qFr. Suppose that $x^1, x^0 \in X$ are such that $x^1 H x^0$, and let $q^1 = g(x^1)$ and $q^0 = g(x^0)$. Then we have $q^1 F q^0$, so that $v(q^1) > v(q^0)$ by virtue of Theorem 5(v). Consequently, we obtain $u(x^1) = v(g(x^1)) > v(g(x^0)) = u(x^0)$; which establishes property (iv).

(b) Let

$$k(q) = \{x : x \in b(q) \cap X \,\&\, \forall y_{y \in b(q) \cap X}\; u(x) \geqq u(y)\}.$$

To prove property (i), it suffices to show that for any $q \in Q$, $h(q) = k(q)$. It is obvious by (a) that $h(q) \subset k(q)$. Now assume $x \in k(q) - h(q)$. Then $y = h(q)$ for some $y \neq x$. This yields yHx, so that $u(y) > u(x)$ by (a). On the other hand, since $x \in k(q)$ and $y \in b(q) \cap X$, it follows that $u(x) \geqq u(y)$; which is a contradiction. Therefore, $h(q) = k(q)$ for all $q \in Q$.

(c) To show property (ii), suppose that $u(x^1) \geqq u(x^0)$, $x^1, x^0 \in X$, $x^1 \neq x^0$, $t \in (0, 1)$, and $x^t = (1 - t)\, x^1 + tx^0 \in X$. Then we find $1 = (1 - t)\, g(x^t)\, x^1 + tg(x^t)\, x^0$. Hence, $g(x^t)\, x^1 \leqq 1$ or $g(x^t)\, x^0 \leqq 1$. If $g(x^t)\, x^1 \leq 1$, then we have $x^t H x^1$, so that $u(x^t) > u(x^1) \geqq u(x^0)$ by (a) and hypothesis. On the other hand, if $g(x^t)\, x^0 \leq 1$, then we have $x^t H x^0$, so that $u(x^t) > u(x^0)$ by (a). In any case, we thus obtain $u(x^t) > u(x^0)$.

(d) Finally, to prove property (iii), let $x^1, x^0 \in X$ be such that $x^1 \geq x^0$. Then, clearly, we have $x^1 H x^0$, which gives $u(x^1) > u(x^0)$ by (a). Thus the proof is complete. Q.E.D.

Comparison of Theorems 5 and 6 indicates the basic duality that exists between the indirect and direct utility functions. First of all, minimizing the indirect utility v of prices q is equivalent to maximizing the direct utility u of commodities x, the identical budget constraint $qx \leq 1$ being imposed in both instances. Furthermore, v is strictly quasi-convex on Q, while u is strictly quasi-concave on X; v is decreasing on Q, while u is increasing on X.

Finally, v represents the indirect revealed favorability relation F on Q, whereas u represents the indirect revealed preference relation H on X.

Remark 2. Although the indirect utility function v is lower semicontinuous (Theorem 5(ii)), the direct utility function u may not be upper semicontinuous under the present assumptions. Symmetry between the two functions appears to break down here. This point is well illustrated by Sonnenschein's Example 3 [20, pp. 274–275], in which the demand function h generated by the indicated direct utility function u clearly satisfies (H), (SF), and (D), but no upper semicontinuous function can generate h. It should be noticed that the range of h, X, is not convex in this example. This once again clarifies the role played by the convexity assumption on the range of the demand function in studying the theory of consumer's demand.

Remark 3. Relation to Richter's results. Richter has shown that if the demand function h satisfies (SP) and its range X possesses a certain convex property, then h is representable on X (see [13, Theorems 12 and 13, pp. 50–51]). Theorem 6 shows that if (SP) is replaced by (SF), such a convexity assumption on X can be entirely eliminated to establish the representability results on X. Also note that Richter's Example 3 [13, p. 47] does not provide a counterexample to Theorem 6. Indeed, the demand function h defined there does not satisfy (SF), although it satisfies (SP) along with (H) and (D).

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Dual Price and Quantity Aggregation

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In his seminal 1959 paper (in *Econometrica*), Gorman developed conditions for price aggregation in the context of income-constrained utility maximization. Price aggregation is defined as the existence of group income allocation functions with price indices and income as arguments. This paper develops and proves propositions about quantity aggregation that are dual to the Gorman results. These dual structures contain interesting asymmetries. The quantity aggregation concepts should be useful in the study of organizational structures characterized by decentralized shadow price (demand price) determination.

1. INTRODUCTION

In his seminal 1959 paper, Gorman [10] developed sufficient conditions[1] for ("perfect") price aggregation in the context of income-constrained utility maximization. Price aggregation is defined as the existence of functions θ^r, $r = 1,..., m$, with images

$$y_r = \theta^r(\Pi^1(P^1),..., \Pi^m(P^m), y), \qquad r = 1,..., m,$$

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[1] The sufficient conditions are also necessary under the assumption of weakly separable preferences (over the consumption space).

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where y is total expenditure, P^r is the vector of prices of goods in the rth group, each Π^r is a positively linearly homogeneous function, and y_r is optimal expenditure on the rth group. Thus, the set of θ^r's constitutes an allocation rule which requires total expenditure and aggregate price indices as informational input. Subsequently, Pollak [20] developed sufficient conditions for a weaker form of price aggregation, defined by the existence of functions $\hat{\theta}^r$, $r = 1,\ldots, m$, such that

$$y_r/y = \hat{\theta}^r(\hat{\Pi}^1(P^1/y),\ldots, \hat{\Pi}^m(P^m/y)), \qquad r = 1,\ldots, m.$$

Interest in these notions of price aggregation is motivated by the consideration of decentralized determination of quantities, given parametric prices and total expenditure.

The attention given to the theory of duality in recent years[2] has reactivated interest in the derivation of demand price, or shadow price, functions. This revival of interest in shadow price determination, together with the recent development of a body of literature on duality and functional structure,[3] prompts us to examine the concepts of, and sufficient conditions for, duals to the above notions of price aggregation.

The price aggregation principles are applicable to the study of decentralized decision making in any organization which allocates expenditures on a set of commodities and/or services subject to parametric prices. They are, however, less applicable to the study of decision making in an organization which is faced with fixed allotments of inputs and a fixed valuation of its output and which is charged with the responsibility of computing shadow prices to be used by planning authorities in adjusting input allotments. Examples of such organizational structures might be found in planned economies. In organizational structures characterized by decentralized shadow price determination, quantity aggregation in the determination of sectoral output evaluations (the y_r's) plays a role analogous to price aggregation in the case of decentralized quantity determination.

Section 2 lays out our notation, some fundamental definitions, and develops the notion of conditional utility functions. Section 3 analyzes the dual structure of price and quantity aggregation, and Section 4 proves some theorems on quantity aggregation. Fairly straightforward alterations of some of these proofs generate alternative (and hopefully simpler) proofs of the Gorman [10] theorems on price aggregation.

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[2] See, for example, [7, 12, 13, 18, 19, 22].

[3] See, for example, [2, 3, 15, 17, 21].

2. SEPARABILITY AND CONDITIONAL UTILITY FUNCTIONS

2.1. *Functional Separability*

Let Ω^n and Ω_+^n be the nonnegative and strictly positive Euclidean n orthants, respectively. Denote commodity bundles by $X = [x_1, \ldots, x_n] \in \Omega^n$, and corresponding price vectors by $P = [p_1, \ldots, p_n] \in \Omega_+^n$. Letting the strictly positive scalar y represent consumer expenditure, $(P/y) \in \Omega_+^n$ is the normalized price vector.

The variable indices of X and P form the set $I = \{1, \ldots, n\}$. Partition I into m subsets or sectors, $\{I^1, \ldots, I^m\}$. Correspondingly, the vectors X and P have decompositions, $X = [X^1, \ldots, X^m]$ and $P = [P^1, \ldots, P^m]$. Similarly, Ω^n and Ω_+^n have Cartesian decompositions $\Omega^n = \Omega^{(1)} \times \cdots \times \Omega^{(m)}$ and $\Omega_+^n = \Omega_+^{(1)} \times \cdots \times \Omega_+^{(m)}$, where, of course, the dimensionality of $\Omega^{(r)}$ and $\Omega_+^{(r)}$ is given by the cardinality of I^r. When the kth good or price is in the rth sector, x_k is a component of $X^r \in \Omega^{(r)}$ and p_k is a component of $P^r \in \Omega_+^{(r)}$.

Let $U: \Omega^n \to \Omega$ be a continuous, semistrictly quasi-concave,[4] nondecreasing utility function, and let $V: \Omega_+^n \to \Omega$, defined by

$$V(P/y) = \operatorname*{Max}_X \{U(X) \mid (P/y) \cdot X \leq 1\},$$

be the corresponding indirect utility function. Given the properties of U, V is necessarily continuous, semistrictly quasi-convex, and nonincreasing.

Partition I into I^r and I^c by letting $I^c = \bigcup_{s \neq r} I^s$, and define the mapping, $\beta^r: \Omega^n \to \mathscr{P}(\Omega^{(r)})$, by

$$\beta^r(X^r, X^c) = \{\hat{X}^r \in \Omega^{(r)} \mid U(\hat{X}^r, X^c) \geq U(X^r, X^c)\}.$$

This correspondence therefore defines a set of points in $\Omega^{(r)}$ for each fixed reference vector (X^r, X^c), such that each point in $\beta^r(X^r, X^c) \times \{X^c\}$ is "no worse than" (X^r, X^c). The set of variables, I^r, is *separable* from the kth variable in U if $\beta^r(X^r, X^c)$ is invariant with respect to the value of the kth variable, x_k. This separability condition, due to Stigum [23], is equivalent to Gorman's [11] condition that "the conditional ordering on $[\Omega^{(r)}]$ is the same for all" values of x_k.

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[4] Semistrict quasi-concavity is defined by $U(X) > U(\hat{X}) \Rightarrow U(\lambda X + (1 - \lambda)\hat{X}) > U(\hat{X})$, $\forall \lambda \in (0, 1)$, $\forall (X, \hat{X}) \in \Omega^n \times \Omega^n$. This assumption implies that upper-level sets are convex (but not necessarily strictly convex). It also precludes "thick" indifference sets, except possibly for an unbounded region of indifference corresponding to a globally maximal level of utility. Semistrict quasi-convexity is defined by reversing the above inequalities.

Similarly, the set I^r is separable from the kth variable in V, $k \notin I^r$, if the mapping, $\alpha^r: \Omega_+{}^n \rightarrow \mathscr{P}(\Omega_+^{(r)})$, defined by

$$\alpha^r\left(\frac{P^r}{y}, \frac{P^c}{y}\right) = \left\{\frac{\hat{P}^r}{y} \in \Omega_+^{(r)} \,\middle|\, V\left(\frac{\hat{P}^r}{y}, \frac{P^c}{y}\right) \geq V\left(\frac{P^r}{y}, \frac{P^c}{y}\right)\right\},$$

is independent of the value of the kth normalized price p_k/y.

Consumer preferences are *directly strongly separable*[5] if every proper subset of the partition $\hat{I} = \{I^1,\ldots, I^m\}$ is separable from its complement in U; i.e., the union of any number of sectors is separable from the variables in the remaining sectors. Preferences are *directly weakly separable* if every sector, I^r, is separable in U from the variables in all the other sectors.

Results of Debreu [5] and Gorman [11] characterize the forms of the utility functions implied by these symmetric structures. If $m > 2$,[6] preferences are directly strongly separable if and only if there exist continuous semistrictly quasi-concave functions $\tilde{U}$, u^1, u^2,..., u^m, such that the utility function image can be written as

$$U(X^1,\ldots, X^m) = \tilde{U}(u^1(X^1) + \cdots + u^m(X^m)),$$

where $\tilde{U}$ is increasing in its single argument and each u^r is nondecreasing in X^r. Of course, U can be normalized so that

$$U(X) = u^1(X^1) + \cdots + u^m(X^m).$$

Preferences are directly weakly separable if and only if there exist continuous, semistrictly quasi-concave functions, $\hat{U}$, u^1,..., u^m, such that the utility function image can be written as

$$U(X^1,\ldots, X^m) = \hat{U}(u^1(X^1),\ldots, u^m(X^m)),$$

where $\hat{U}$ is increasing in each of its m arguments and each u^r is nondecreasing in X^r. If a function is weakly separable and each of the category functions is homothetic, the function is said to be *homothetically separable*. If the function is homothetically separable, each category felicity function may be trivially normalized to be positively linearly homogeneous (PLH).

Indirect weak and strong separability are defined analogously to direct weak and strong separability by replacing the direct utility function U with the indirect utility function V and X with P/y. Thus, if $m > 2$, indirect strong separability is equivalent to the existence of continuous, semistrictly quasi-concave, nondecreasing functions, $v^1,\dots, v^m$, and a continuous, decreasing function $\hat{V}$ such that

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[5] That is, strongly separable "in the indicated partition of I." This phrase is implicitly included in all of our discussion of separable structures.

[6] If $m = 2$, weak and strong separability coincide and the following additive representation does not go through.

$$V(P/y) = \hat{V}\left(\sum_{r=1}^{m} v^r(P^r/y)\right).^7$$

Indirect weak separability is equivalent to the existence of continuous, semistrictly quasi-concave, nondecreasing functions $v^1,\dots, v^m$, and a continuous, semistrictly quasi-convex, decreasing function $\hat{V}$, such that

$$V(P/y) = \hat{V}(v^1(P^1/y),\dots, v^m(P^m/y)).$$

The notion of indirect homothetic separability is analogous to direct homothetic separability, discussed above.

In general, direct and indirect separability do not imply one another. However, U is homothetically separable if and only if V is negatively homothetically separable (see [17]).

2.2 *Conditional Utility Functions*

The conditional indirect utility function, H, is defined by

$$H({}_1y, P) = \{\operatorname*{Max}_X U(X) \mid P^r \cdot X^r \leqslant y_r\,,\ r = 1,\dots, m\}, \tag{1}$$

where ${}_1y = [y_1,\dots, y_m]$ is a vector of (undetermined) "category incomes." The conditional utility function inherits the properties of U in ${}_1y$ and the properties of V in P^r, $r = 1,\dots, m$. Furthermore, H is continuous in $({}_1y, P)$ and homogeneous of degree zero (HDO) in each pair (y_r, P^r). Finally, to any H with these properties there corresponds a utility function satisfying the above stated regularity conditions. (See [6, 8] for proofs of these duality results.)

Just as H is dual to U, we can define the conditional direct utility function, G, which is dual to $\overset{*}{V}$, the extension by lower semicontinuity of the indirect utility function to Ω^n, by[8]

$$G({}_1y/y, X) = \operatorname*{Min}_{P/y} \{\overset{*}{V}(P/y) \mid (P^r/y) \cdot X^r \leqslant (y_r/y),\ r = 1,\dots, m\}. \tag{2}$$

The conditional direct utility function inherits the properties of V in ${}_1y/y$ and the properties of U in X^r, $r = 1,\dots, m$. In addition, G is jointly continuous in

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[7] Obviously, this representation could be renormalized so that $\hat{V}$ is increasing and each v^r is nonincreasing.

[8] This extension is required in order to compactify the constraint set in the optimization problem that defines G. See [7] for a discussion of extension by lower continuity of the indirect utility function.

$({}_1y/y, X)$ and homogeneous of degree zero in each pair $(y_r/y, X^r)$. The proofs of these assertions parallel the arguments of Diewert [6] and Epstein [8] for the properties of H.

In order to examine separability properties of the direct and indirect conditional utility functions, it is useful to introduce the notation $R = \{1,..., m\}$ for the set of group (and group expenditure) indices. Further, $\widehat{RI} \equiv \{(1, I^1),..., (m, I^m)\}$ is a partition of $R \cup \hat{I}$ that is induced by the partition $\hat{I}$ of I. Separability of (r, I^r) from a variable in its complement (relative to $R \cup \hat{I}$) is defined analogously to separability in the direct and indirect utility functions. Structures induced by separability—all with respect to the partition $\widehat{RI}$—in the direct and indirect conditional utility functions are as follows:

Weak separability of H.

$$H({}_1y, P) = \hat{H}(h^1(y_1, P^1),..., h^m(y_m, P^m)).$$

Strong separability of H $(m > 2)$.

$$H({}_1y, P) = \tilde{H}\left(\sum_{r=1}^{m} h^r(y_r, P^r)\right).$$

Weak separability of G.

$$G({}_1y/y, X) = \hat{G}(g^1(y_1/y, X^1),..., g^m(y_m/y, X^m)).$$

Strong separability of G $(m > 2)$.

$$G({}_1y/y, X) = \tilde{G}\left(\sum_{r=1}^{m} g^r(y_r/y, X^r)\right).$$

The h^r and g^r $(r = 1,..., m)$ functions in the above representations inherit the properties of H and G, respectively, and $\hat{H}$, $\tilde{H}$, $\hat{G}$, and $\tilde{G}$ are continuous, semistrictly quasi-concave, and increasing.

THEOREM 1. *Suppose that U is continuous, semistrictly quasi-concave, and nondecreasing. The indirect (respectively direct) conditional utility function is weakly (respectively strongly) separable in the partition* $\widehat{RI}$ *if and only if the direct (indirect) utility function is weakly (strongly) separable in the corresponding partition* $\hat{I}$.

Proof. If U is weakly separable in $\hat{I}$,

$$\begin{aligned} H({}_1y, P) &= \operatorname*{Max}_{X} \{\breve{U}(u^1(X^1),..., u^m(X^m)) \mid P^r \cdot X^r \leqslant y_r, r = 1,..., m\} \\ &= \breve{U}(\operatorname*{Max}_{X^1} \{u^1(X^1) \mid P^1 \cdot X^1 \leqslant y_1\},..., \operatorname*{Max}_{X^m} \{u^m(X^m) \mid P^m \cdot X^m \leqslant y_m\}) \\ &= \breve{U}(h^1(y_1, P^1),..., h^m(y_m, P^m)), \end{aligned}$$

where we have exploited (in the decomposition of the optimization problem) the strict monotonicity property of $\hat{U}$. The construction and the properties of $\hat{U} = \hat{H}$ and u^r, $r = 1,\ldots, m$, imply that H is weakly separable in $\widehat{RI}$.

To prove the converse, let

$$\begin{aligned}&\hat{H}(u^1(X^1),\ldots, u^m(X^m))\\&\quad = \operatorname*{Min}_{P} \{\hat{H}(h^1(1, P^1),\ldots, h^m(1, P^m)) \mid P^r \cdot X^r \leq 1,\ r = 1,\ldots, m\},\end{aligned}$$

where, since $\hat{H}$ is increasing,

$$u^r(X^r) = \operatorname*{Min}_{P^r} \{h^r(1, P^r) \mid P^r \cdot X^r \leq 1\}, \qquad r = 1,\ldots, m.[9]$$

However, by duality,

$$h^r(y_r, P^r) = \operatorname*{Max}_{X^r} \{u^r(X^r) \mid P^r \cdot X^r \leq y_r\}, \qquad r = 1,\ldots, m,$$

so that

$$\begin{aligned}&\hat{H}(h^1(y_1, P^1),\ldots, h^m(y_m, P^m))\\&\quad = \operatorname*{Max}_{X} \{\hat{H}(u^1(X^1),\ldots, u^m(X^m)) \mid P^r \cdot X^r \leq y_r,\ r = 1,\ldots, m\}.\end{aligned}$$

Since, by definition,

$$H({}_1y, P) = \operatorname*{Max}_{X} \{U(X) \mid P^r \cdot X^r \leq y_r,\ r = 1,\ldots, m\},$$

it follows that

$$U(X) = \hat{H}(u^1(X^1),\ldots, u^m(X^m)).$$

Moreover, the properties of $\hat{H}$ and u^r, $r = 1,\ldots, m$, imply that U is weakly separable in $\hat{I}$.

Analogous constructions establish the equivalence of strong separability of U and H and weak and strong separability of V and G. ▮

The direct and indirect conditional utility functions and their properties are very instrumental in proving the theorems on price and quantity aggregation. For the same purpose, it will be useful to consider additional restrictions on these functions. If the functions h^r, $r = 1,\ldots, m$, satisfy

$$h^r(y_r, P^r) = \Psi^r\left(\frac{y_r}{\Pi^r(P^r)}\right) + \Lambda^r(P^r), \qquad \text{(GGPF)}$$

where Π^r is PLH, Λ^r is HDO, and Ψ^r is increasing, we say that the aggregator function u^r has the *generalized Gorman polar form* [10].[10] Note that the properties of Π^r and Λ^r imply that

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[9] The following duality argument is carried out at unit category income rather than by normalizing prices by y_r (exploiting the homogeneity property of h^r), since y_r could be zero, in which case P^r/y_r is undefined.

$$h^r(y_r/y, P^r/y) = \Psi^r\left(\frac{y_r/y}{\Pi^r(P^r/y)}\right) + \Lambda^r(P^r/y).$$

Similarly, treating P^r/y as dual to X^r and y_r/y as self-dual we say that the indirect utility aggregator v^r has the *generalized Gorman form* if

$$g^r(y_r/y, X^r) = \Xi^r\left(\frac{y_r/y}{\Gamma^r(X^r)}\right) + \chi^r(X^r), \qquad \text{(GGF)}$$

where Γ^r is PLH, χ^r is HDO, and Ξ^r is decreasing. We refer to (GGF) and (GGPF) as "structural duals," since they are obtained by imposing the same structure on the dual functions G and H (with respect to the dual variables $(X, {}_1y/y)$ and $(P/y, {}_1y/y)$. They are *not* duals in the sense of representing the same class of preferences; the Gorman polar form structure does not have the same implications for consumer preferences as does the Gorman form structure.

3. Price and Quantity Aggregation

Gorman [10] defined perfect price aggregation as the existence of income allocation functions[11] θ^r and PLH price aggregates Π^r, $r = 1,\dots, m$, such that

$$y_r = \theta^r(\Pi^1(P^1),\dots, \Pi^m(P^m), y), \qquad r = 1,\dots, m. \qquad \text{(SPA)}$$

Clearly, each θ^r is PLH in P and y so that (SPA) implies

$$\begin{aligned} y_r/y &= \hat{\theta}^r(\Pi^1(P^1/y),\dots, \Pi^m(P^m/y)) \\ &= \theta^r(\Pi^1(P^1/y),\dots, \Pi^m(P^m/y), 1), \qquad r = 1,\dots, m. \end{aligned} \qquad \text{(WPA)}$$

On the other hand, (WPA) does not imply (SPA), since $\hat{\theta}^r$ in general has no homogeneity properties in its arguments (although each Π^r remains PLH). For this reason, we refer to (SPA) as *strong price aggregation* and to (WPA) as *weak price aggregation.*

If, in addition to the existence of the above price indices, there exist PLH quantity indices $u^1,\dots, u^m$, such that

$$\Pi^r(P^r) \cdot u^r(X^r) = y_r, \qquad r = 1,\dots, m, \qquad \text{(APA)}$$

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[10] The ordinary Gorman polar form—which has interesting implications for aggregation across agents—is generated by letting ψ be the identity function [9].

[11] Under our regularity conditions, θ^r might be a correspondence. This causes no problems in what follows.

holds for the optimal triple (P^r, X^r, y_r), the preferences are characterized by *additive price aggregation.* If this condition is satisfied, it is possible to formulate a first-stage optimization problem in which the consumer maximizes utility with respect to the composite commodities, $u^1, \ldots, u^m$, subject to the above budget constraint in aggregate prices and commodities.

Known necessary and sufficient conditions for the above concepts are summarized in the following theorem.

THEOREM 2. *If U is continuous, semistrictly quasi-concave, and nondecreasing, the following results hold for a partition $\hat{I}$ with more than two groups:*

(i) *Homothetic separability implies strong (and hence weak) price aggregation* [10].

(ii) *Direct strong separability with the generalized Gorman polar form implies strong (and hence weak) price aggregation* [10].

(iii) *If U is differentiable and weakly separable and if not precisely $m - 1$ of the aggregator functions are homothetic, strong price aggregation implies either homothetic separability or direct strong separability with the generalized Gorman polar form or some hybrid combination of these two structures; i.e.,*

$$U(X) = \bar{U}\left(\sum_{r=1}^{d} u^r(X^r) + \hat{U}(u^{d+1}(X^{d+1}), \ldots, u^m(X^m))\right),$$

where each u^r, $r = 1, \ldots, d$ $(d \neq 1)$, has the generalized Gorman polar form and each u^r, $r = d + 1, \ldots, m$, is homothetic [10].

(iv) *If V is differentiable, indirect strong separability implies weak price aggregation* [20].

(v) *Homothetic separability is sufficient for additive price aggregation* [10]. *If U is weakly separable and differentiable, homothetic separability is necessary for additive price aggregation* [1].

Although proof of this proposition is unnecessary here, it is instructive to examine the derivation of income allocation functions when each of the sufficient conditions in (i) and (ii), is, alternatively, satisfied. Suppose, therefore, that the direct utility function is strongly separable with generalized Gorman polar forms. By Theorem 1, the indirect conditional utility function can be written as

$$H({}_1y, P) = \sum_{r=1}^{m} \left[\Psi^r\left(\frac{y_r}{\Pi^r(P^r)}\right) + \Lambda^r(P^r)\right].$$

Solving

$$\operatorname*{Max}_{{}_1y} H({}_1y, P) \text{ s.t. } \sum_{r=1}^{m} y_r = y$$

clearly yields income allocation correspondences of the form

$$y_r = \theta^r(\Pi^1(P^1),\dots, \Pi^m(P^m), y), \qquad r = 1,\dots, m, \tag{SPA}$$

which can, of course, be rewritten as

$$y_r/y = \hat{\theta}^r(\Pi^1(P^1/y),\dots, \Pi^m(P^m/y)), \qquad r = 1,\dots, m. \tag{WPA}$$

The "structural" dual to the above conditional indirect utility function is

$$G({}_1y/y, X) = \sum_{r=1}^{m} \left[\Xi^r\left(\frac{y_r/y}{\Gamma^r(X^r)}\right) + \chi^r(X^r)\right].$$

Solving

$$\operatorname*{Max}_{{}_1y/y} G({}_1y/y, X) \text{ s.t. } \sum_{r=1}^{m} (y_r/y) = 1$$

yields

$$y_r/y = \hat{\eta}^r(\Gamma^1(X^1),\dots, \Gamma^m(X^m)), \qquad r = 1,\dots, m. \tag{SQA}$$

The homogeneity property of Γ^r, $r = 1,\dots, m$, allows us to write

$$\begin{aligned} y_r &= y\hat{\eta}^r(y\Gamma^1(X^1/y),\dots, y\Gamma^m(X^m/y)) \\ &= \eta^r(\Gamma^1(X^1/y),\dots, \Gamma^m(X^m/y), y), \qquad r = 1,\dots, m. \end{aligned} \tag{WQA}$$

Note, however, that, as η^r generally has no homogeneity properties, it is not generally possible to go from (WQA) to (SQA). It is natural, therefore, to define (SQA) as *strong quantity aggregation* and (WQA) as *weak quantity aggregation.*

It is interesting to note that X^r is dual to P^r/y, X^r/y is dual to P^r, and y_r/y is self-dual. Hence, *strong quantity aggregation* is structurally dual to *weak price aggregation* and *weak quantity aggregation* is structurally dual to *strong price aggregation.* This dual structure is further explicated upon examination of the cases where the Λ^r and the χ^r function images vanish, but the assumption of strong separability is relaxed to weak separability. The indirect conditional utility function then has image

$$H({}_1y, P) = \bar{H}\left(h^1\left(\frac{y_1}{\Pi^1(P^1)}\right),\dots, h^m\left(\frac{y_m}{\Pi^m(P^m)}\right)\right).$$

The structural dual is

$$G({}_1y/y, X) = \bar{G}\left(g^1\left(\frac{y_1/y}{\Gamma^1(X^1)}\right),\dots, g^m\left(\frac{y_m/y}{\Gamma^m(X^m)}\right)\right).$$

It is clear from this representation that the corresponding indirect utility aggregators, v^r, $r = 1,\dots, m$, and direct utility aggregators, u^r, $r = 1,\dots, m$, are homothetic. It is also straightforward to show that these structures imply strong price and quantity aggregation, respectively. Moreover, in the appropriate normalizations

$$\frac{y_r}{\Pi^r(P^r)} = \max_{X^r}\,\{u^r(X^r) \mid P^r\cdot X^r \leqslant y_r\} \qquad \text{(APA)}$$

or

$$y_r = \Pi^r(P^r)\cdot u^r(\overset{*}{X}{}^r),$$

where $\overset{*}{X}{}^r$ solves the maximization problem; letting $\overset{*}{P}{}^r/y$ solve the minimization problem,

$$\frac{y_r/y}{\Gamma^r(X^r)} = \min_{P^r/y}\left\{v^r(P^r/y)\,\middle|\,\frac{P^r}{y}\cdot X^r \leqslant \frac{y_r}{y}\right\},$$

results in

$$y_r/y = v^r(\overset{*}{P}{}^r/y)\,\Gamma^r(X^r), \qquad \text{(AQA)}$$

which implies

$$y_r = v^r(\overset{*}{P}{}^r)\,\Gamma^r(X^r).$$

Duality requires that $(\overset{*}{P}{}^r, \overset{*}{X}{}^r)$ be optimal pairs. It is therefore true that

$$\Gamma^r(X^r) = u^r(X^r)$$

and

$$\Pi^r(P^r) = v^r(P^r).$$

Thus, in the case of homothetic separability, the income allocation correspondences

$$y_r = \theta^r(\Pi^1(P^1),\dots, \Pi^m(P^m), y) \qquad \text{(SPA)}$$

and

$$y^r = y\eta^r(\Gamma^1(X^1/y),\dots, \Gamma^m(X^m/y)) \qquad \text{(SQA)}$$

can be generated by solving the dual problems

$$\max_{\Gamma^1(X^1),\dots,\Gamma^m(X^m)} U(\Gamma^1(X^1),\dots, \Gamma^m(X^m))\ \text{s.t.} \sum_{r=1}^{m} \Pi^r(P^r)\cdot\Gamma^r(X^r) \leqslant y$$

and

$$\operatorname*{Min}_{\Pi^1(P),\dots,\Pi^m(P^m)} V(\Pi^1(P^1),\dots,\Pi^m(P^m)) \text{ s.t. } \sum_{r=1}^{m} \Pi^r(P^r)\cdot\Gamma^r(X^r) \leq y.$$

The solutions which are obtained from these dual problems are

$$\Gamma^r(X^r) = \tilde{\theta}^r(\Pi^1(P^1),\dots,\Pi^m(P^m), y), \qquad r = 1,\dots, m,$$

and

$$\Pi^r(P^r) = \tilde{\eta}^r(\Gamma^1(X^1),\dots,\Gamma^m(X^m), y), \qquad r = 1,\dots, m,$$

respectively. Then

$$\theta^r(\Pi^1(P^1),\dots,\Pi^m(P^m), y) = \Pi^r(P^r)\cdot\tilde{\theta}^r(\Pi^1(P^1),\dots,\Pi^m(P^m), y), \qquad r = 1,\dots, m,$$

and

$$\begin{aligned} &y\cdot\eta^r(\Gamma^1(X^1/y),\dots,\Gamma^m(X^m/y)) \\ &\quad = y\cdot\Gamma^r(X^r/y)\cdot\tilde{\eta}^r(\Gamma^1(X^1),\dots,\Gamma^m(X^m), y) \\ &\quad = y\cdot\Gamma^r(X^r/y)\cdot\tilde{\eta}^r(\Gamma^1(X^1/y),\dots,\Gamma^m(X^m/y), 1) \qquad r = 1,\dots, m. \end{aligned}$$

The last equality holds because each $\tilde{\eta}^r$ is HDO and each Γ^r is PLH.

Recalling that (APA) is additive price aggregation, it is natural to call (AQA) additive quantity aggregation; however, because of the dual structure, there is no distinction between additive price aggregation and additive quantity aggregation. This equivalence follows from the fact that direct homothetic separability is necessary and sufficient for indirect homothetic separability.

In the next section, we prove a quantity aggregation theorem which is dual to Theorem 2 on price aggregation.

4. A QUANTITY AGGREGATION THEOREM

The dual structure of price and quantity aggregation outlined in Section 3 suggests the following quantity aggregation theorem that is dual[12] to Theorem 2:

THEOREM 3. *If V is continuous, semistrictly quasi-convex, and nonincreasing in P/y, the following results hold for a partition $\hat{I}$ with more than two groups:*

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[12] Actually the reader might notice some asymmetries between Theorems 2 and 3 in the differentiability assumptions of parts (iii) and (v). This is only because different authors have used different assumptions and methods of proof. However, the theorems could have been stated with a set of symmetric assumptions and are indeed dual results.

(i) *Homothetic separability implies strong (and hence weak) quantity aggregation.*

(ii) *Indirect strong separability with the generalized Gorman form implies strong (and hence weak) quantity aggregation.*

(iii) *If G is differentiable, V is weakly separable, and not precisely $m - 1$ of the aggregator functions are homothetic, strong quantity aggregation implies either homothetic separability or indirect strong separability with the generalized Gorman form or some hybrid combination of these two structures; i.e.,*

$$V(P/y) = \bar{V}\left(\sum_{r=1}^{d} v^r(P^r/y) + \hat{V}(v^{d+1}(P^{d+1}/y),\dots, v^m(P^m/y))\right),$$

where each v^r, $r = 1,\dots, d$ $(d \neq 1)$, has the generalized Gorman form and each v^r, $r = d+1,\dots, m$, is homothetic.

(iv) *If U is differentiable, direct strong separability implies weak quantity aggregation.*

(v) *Homothetic separability is sufficient for additive quantity aggregation. If V is weakly separable, homothetic separability is necessary for additive quantity aggregation.*

In proving this theorem, it is useful to first prove a lemma (which may have other interesting applications).

LEMMA. *Suppose the choice function*

$$\phi: \Omega_+^{\rho+1} \times \Omega^m \to \Omega^\rho$$

with image

$$Z = \phi(B, c, \zeta(X)), \qquad B = [b_1,\dots, b_\rho] \in \Omega_+{}^\rho, \qquad c \in \Omega_+$$

(where $\zeta(X)$ is the m-tuple of images $(\zeta^1(X^1),\dots, \zeta^m(X^m))$) solves the optimization problem

$$\max_Z f(Z, X) \text{ s.t. } B \cdot Z = c,$$

where the objective function

$$f: \Omega^\rho \times \Omega^n \to \Omega$$

is semistrictly quasi-concave, continuous, and nondecreasing in Z. Suppose also that the (indirect) function $v: \Omega_+^{\rho+1} \times \Omega^n \to \Omega$ defined by

$$v(B, c, X) = f(\phi(B, c, \zeta(X)), X)$$

is continuously differentiable in B and c. Then there exist two functions,

$$g\colon \Omega^{p} \times \Omega^{m} \to \Omega$$

and

$$h\colon \Omega^{n} \to \Omega,$$

such that

$$f(Z, X) = g(Z, \zeta(X)) + h(X).$$

Proof. Roy's theorem,

$$\phi_i(B, c, \zeta(X)) = -\frac{\partial v(B, c, X)/\partial b_i}{\partial v(B, c, X)/\partial c}, \qquad i = 1,\ldots, p,$$

implies that gradients of level sets of v have components

$$\frac{\partial v(B, c, X)/\partial b_i}{\partial v(B, c, X)/\partial b_j} = \frac{\phi_i(B, c, \zeta(X))}{\phi_j(B, c, \zeta(X))}.$$

Integration over B and c yields

$$v(B, c, X) = w(B, c, \zeta(X)) + h(X).$$

Noting that v is HDO in B and c, we can write

$$v(B, c, X) = \hat{v}(B/c, X)$$

and

$$w(B, c, \zeta(X)) = \hat{w}(B/c, \zeta(X)).$$

Finally,

$$\begin{aligned} f(Z, X) &= \operatorname*{Min}_{B/c} \{\hat{v}(B/c, X) \mid (B/c) \cdot Z \leqslant 1\} \\ &= \operatorname*{Min}_{B/c} \{\hat{w}(B/c, \zeta(X)) + h(X) \mid (B/c) \cdot Z \leqslant 1\} \\ &= g(Z, \zeta(X)) + h(X). \quad \blacksquare \end{aligned}$$

Proof of Theorem 2. Proofs of parts (i) and (ii) are contained in the discussion at the end of Section 3.

Proof of (iii). We are given

$$y_r/y = \hat{\eta}^r(\Gamma^1(X^1),\ldots, \Gamma^m(X^m)), \qquad r = 1,\ldots, m, \tag{SQA}$$

where each Γ^r is PLH in X^r. (SQA) solves the problem,

$$\mathop{\mathrm{Min}}_{{}_1y,\,y} G({}_1y/y,\, X) \text{ s.t. } \sum_{r=1}^{m} y_r/y = 1,$$

where G is the conditional direct utility function defined earlier. Hence, by the lemma there exist functions, say g and χ, such that G can be written as

$$G({}_1y/y,\, X) = g(\Gamma^1(X^1),\ldots,\, \Gamma^m(X^m),\, {}_1y/y) + \chi(X). \tag{2}$$

The homogeneity properties of G, $\Gamma^1,\ldots, \Gamma^m$ imply that g is HDO in the pairs $(\Gamma^r(X^r),\, y_r/y)$, $r = 1,\ldots, m$, and that χ is HDO in each X^r (and hence in X). Since V is assumed to be weakly separable in the partition $\hat{I}$, G is weakly separable in the corresponding partition $\widehat{RI}$. Hence, there exist functions $\hat{G}$, $g^1,\ldots, g^m$ such that G can be written as

$$G({}_1y/y,\, X) = \hat{G}(g^1(y_1/y,\, X^1),\ldots,\, g^m(y_m/y,\, X^m)). \tag{3}$$

Since each category function, g^r, mirrors the preordering induced by G (see [11]) we may let

$$g^r(y_r/y,\, X^r) = G(O^{(1)},\ldots,\, y_r/y,\ldots,\, O^{(m)},\, O^{[1]},\ldots\, X^r,\ldots,\, O^{[m]}),$$

where $O^{(s)}$ is an arbitrary reference value of y_s/y and $O^{[s]}$ is an arbitrary reference vector value of X^s, $s = 1,\ldots, m$ $(s \neq r)$. (That is, the image of $(y_r/y,\, X^r)$ by g^r equals the image of $({}_1y/y,\, X)$ by G when $X^s = O^{[s]}$ and $y_s/y = O^{(s)}$ for all $s \neq r$.) Letting $X^s = O^{[s]}$ and $y_s/y = O^{(s)}$ for all $s \neq r$ in (2) gives

$$\begin{aligned} g^r(y_r/y,\, X^r) &= g(\Gamma^1(O^{[1]}),\ldots,\, \Gamma^r(X^r),\ldots,\, \Gamma^m(O^{[m]}),\, O^{(1)},\ldots,\, y_r/y,\ldots\, O^{(m)}) \\ &\quad + \chi(O^{[1]},\ldots,\, X^r,\ldots,\, O^{[m]}). \end{aligned} \tag{4}$$

The homogeneity properties of g and χ imply that (4) can be written as

$$g^r(y_r/y,\, X^r) = \Xi^r\left(\frac{y_r/y}{\Gamma^r(X^r)}\right) + \chi^r(X^r), \qquad r = 1,\ldots, m, \tag{5}$$

where χ^r is HDO in X^r and Ξ^r is a decreasing function of its single argument. Substituting (5) into (3) gives

$$G({}_1y/y,\, X) = \hat{G}\left(\Xi^1\left(\frac{y_1/y}{\Gamma^1(X^1)}\right) + \chi^1(X^1),\ldots,\, \Xi^m\left(\frac{y_m/y}{\Gamma^m(X^m)}\right) + \chi^m(X^m)\right). \tag{6}$$

Suppose that $\chi^r(X^r) = 0$, $r = 1,\ldots, m$. In this case, there is a representation of $V(P/y)$, say

$$V(P/y) = \tilde{V}(\tilde{v}^1(P^1/y),\ldots, \tilde{v}^m(P^m/y)),^{13}$$

such that

$$-\frac{y_r/y}{\Gamma^r(X^r)} = \min_{P^r} \{\tilde{v}^r(P^r/y) \mid (P^r/y) \cdot X^r \leq y_r/y\},$$

which implies that each $\tilde{v}^r$ is PLH in P^r/y.

By the assumption of the theorem, if $\chi^r(X^r) \neq 0$ for some r, then $\chi^r(X^r) \neq 0$ for d groups where $d \geq 1$. Without loss of generality, we may assume that $\chi^r(X^r) \neq 0$ for $r = 1,\ldots, d$ $(d \geq 1)$ and $\chi^r(X^r) \neq 0$ for $r = d+1,\ldots, m$.

Minimizing (6) subject to

$$\sum_{r=1}^{m} y_r/y = 1$$

yields first-order conditions

$$\frac{\hat{G}_r \cdot (\Xi^r)' \cdot \Gamma^s(X^s)}{\hat{G}_s \cdot (\Xi^s)' \cdot \Gamma^r(X^r)} = 1, \qquad r, s = 1,\ldots, m, \tag{7}$$

where $\hat{G}_r$ is the partial derivative of $\hat{G}$ with respect to its rth argument and $(\Xi^r)'$ is the derivative of Ξ^r with respect to its single argument. (SQA) implies that the ratios in (7) are independent of $\chi^r(X^r)$, $r = 1,\ldots, m$. Consequently, $\hat{G}_r/\hat{G}_s$ is independent of $\Xi^r((y_r/y)/\Gamma^r(X^r)) + \chi^r(X^r)$ for all $r \in \{1,\ldots, d\}$. Thus, $\{r, \{d+1,\ldots, m\}\}$ and $\{s, \{d+1,\ldots, m\}\}$ are separable from their complements in $\{1,\ldots, d, \{d+1,\ldots, m\}\}$ for all r, s in $\{1,\ldots, d\}$. The basic theorem of Gorman [11] then implies that the symmetric differences $\{r, s\} \subseteq \{1,\ldots, d\}$ of these intersecting separable sets are themselves separable from their complements in $\{1,\ldots, d, d+1\}$. Therefore $\hat{G}$ is strongly separable with respect to the partition $\{\{1\},\ldots, \{d\}, \{d+1,\ldots, m\}\}$; hence

$$\begin{aligned} G({}_1y/y, X) = \overset{*}{G}\Bigg(&\sum_{r=1}^{d} \Xi^r\left(-\frac{y_r/y}{\Gamma^r(X^r)}\right) + \chi^r(X^r)\Bigg) \\ &+ \hat{G}\left(\Xi^{d+1}\left(\frac{y_{d+1}/y}{\Gamma^{d+1}(X^{d+1})}\right),\ldots, \Xi^m\left(\frac{y_m/y}{\Gamma^m(X^m)}\right)\right). \end{aligned}$$

A construction analogous to that in Theorem 1 establishes the required structure.

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[13] In particular, each $\tilde{v}^r$ satisfies $\Xi^r(\tilde{v}^r(P^r/y)) = v^r(P^r/y)$, $r = 1,\ldots, m$.

Proof of (iv). Applying Wold's theorem,

$$\frac{p_i}{y} = \frac{\partial U(X)/\partial x_i}{\sum_{i\in I} (\partial U(X)/\partial x_i) \cdot x_i},$$

to

$$U(X) = \hat{U}\left(\sum_{r\in R} u^r(X^r)\right),$$

yields

$$\frac{p_i}{y} = \frac{\partial u^r(X^r)/\partial x_i}{\sum_{r\in R} \sum_{i\in I^r} (\partial u^r(X^r)/\partial x_i) \cdot x_i}, \qquad i \in I^r.$$

Multiplying both sides by x_i and summing over $i \in I^r$ yields

$$\frac{y_r}{y} = \frac{\Gamma^r(X^r)}{\sum_{r\in R} \Gamma^r(X^r)}, \qquad r = 1,..., m,$$

where

$$\Gamma^r(X^r) = \sum_{i\in I^r} \frac{\partial u^r(X^r)}{\partial x_i} \cdot x_i, \qquad r = 1,..., m.$$

Proof of (v). Sufficiency of homothetic separability follows immediately from the discussion at the end of Section 3. To prove necessity, suppose that we are given price and quantity indices Π^r and Γ^r such that, for optimal triples $(\overset{*}{P}, \overset{*}{X}, {}_1\overset{*}{y})$,

$$\Pi^r(\overset{*}{P}{}^r) \cdot \Gamma^r(\overset{*}{X}{}^r) = \overset{*}{y}_r, \qquad r = 1,..., m. \tag{8}$$

Weak separability of V implies "weak decentralizability" (see [4, 16]); that is,

$$\overset{*}{P}{}^r/y = \phi^r(\overset{*}{X}{}^r, \overset{*}{y}_r/y) \tag{9}$$

solves

$$\underset{P^r/y}{\text{Min}}\ v^r(P^r/y) \text{ s.t. } (P^r/y) \cdot \overset{*}{X}{}^r \leqslant (\overset{*}{y}_r/y)$$

for all r. Substituting (9) into (8) yields

$$\overset{*}{y}_r/\Gamma^r(\overset{*}{X}{}^r) = \Pi^r(y \cdot \phi^r(\overset{*}{X}{}^r, \overset{*}{y}_r/y)).$$

This identity indicates that the composition $\Pi^r \circ \phi^r$ is PLH in $\overset{*}{y}_r/y$ for all r. Also, as ϕ^r is HDO,

$$\begin{aligned} \lambda\overset{*}{y}_r/\Gamma^r(\lambda\overset{*}{X}{}^r) &= \Pi^r(y \cdot \phi^r(\lambda\overset{*}{X}{}^r, \lambda\overset{*}{y}_r/y)) \\ &= \Pi^r(y \cdot \phi^r(\overset{*}{X}{}^r, \overset{*}{y}_r/y)), \qquad \forall\lambda > 0, \qquad r = 1,..., m, \end{aligned}$$

which proves that Γ^r is PLH for all r. This in turn implies that $\Pi^r \circ \phi^r$ is NLH in $\overset{*}{X}{}^r$ for all r. Next, note that monotonicity of v^r implies that

$$\overset{*}{y}_r = \phi^r(\overset{*}{X}{}^r, \overset{*}{y}_r/y) \cdot \overset{*}{X}{}^r, \qquad r = 1,..., m,$$

so that, because of the homogeneity property of Π^r, ϕ^r and Γ^r,

$$\phi^r(\lambda\dot{X}^r, \dot{y}_r/y) \cdot \lambda\dot{X}^r = \lambda^{-1}\Pi^r(\phi^r(\dot{X}^r, \dot{y}_r/y)) \cdot \lambda\Gamma^r(\dot{X}^r), \qquad r = 1,\ldots, m.$$

It follows that ϕ^r is NLH in $\dot{X}^r$, implying that Π^r is PLH and v^r is negatively homothetic for all r. ▮

It is worth noting that the necessity proof of (v) actually proves a result that is stronger than "additive price aggregation implies homothetic separability." That is, the existence of price and quantity indices which simply "add up"—but are not necessarily PLH—implies homothetic separability and positive linear homogeneity of the indices.

5. CONCLUSION

In conclusion, we present a table which summarizes the known implications between the different aggregation concepts and separability conditions.

Dual Structure of Price and Quantity Aggregation

Variables		Aggregation Concepts	
X	P/y	SQA	WPA
X/y	P	WQA	SPA
(y_r, y)	(y_r, y)	AQA	APA

Implications

HS ∨ (ISS ∧ GGF) ⇒ SQA
SQA ∧ IWS ⇒ HS ∨ (ISS ∧ GGF)
DSS ⇒ WQA
SQA ⇒ WQA
HS ⇒ AQA
AQA ∧ DWS ⇒ HS

HS ∨ (DSS ∧ GGPF) ⇒ SPA
SPA ∧ DWS ⇒ HS ∨ (DSS ∧ GGPF)
ISS ⇒ WPA
SPA ⇒ WPA
HS ⇒ APA
APA ∧ IWS ⇒ HS

Glossary

SQA	Strong quantity aggregation
WQA	Weak quantity aggregation
AQA	Additive quantity aggregation
SPA	Strong price aggregation
WPA	Weak price aggregation
APA	Additive price aggregation
DWS	Direct weak separability
DSS	Direct strong separability
IWS	Indirect weak separability
ISS	Indirect strong separability
HS	Homothetic separability
GGPF	Generalized Gorman polar form
GGF	Generalized Gorman form
⇒	Conditional
∧	Conjunction
∨	Alternation

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Transitivity of Preferences on a Smooth Manifold of Alternatives

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INTRODUCTION

Arrow's theorem [1] has shown that a collective choice rule, which maps individual preferences to a social preference, cannot satisfy the properties of independence of irrelevant alternatives, Pareto optimality, and nondictatorship, and yet guarantee transitive social preferences for any configuration of individual preferences.

A number of writers, including Sen [9], Sen and Pattanaik [11], and Inada [4], have sought restrictions or exclusion principles on the pattern of individual preferences which are sufficient to guarantee transitivity of the method of majority decision making.

To be more explicit, let W be the set of alternatives and let θ be the set of all distinct preference orderings on W. An exclusion principle, E say, determines a class $S(E)$ of subsets of θ. If, for any combination of individual preferences, all of which belong to some set in $S(E)$, a collective choice rule f satisfies a particular equilibrium condition, then say the exclusion principle E is sufficient for this equilibrium property to hold for f. Suppose now that we are informed about the preferences of a set M of individuals belonging to some collectivity N. Say that the preferences of the individuals in M are incompatible with the principle E if no combination of preferences for the set M can belong to some set in $S(E)$. Say the exclusion principle E is necessary for an equilibrium condition on f, if, whenever there is set M whose preferences are incompatible with E, there exists some set of preferences for a group of individuals $N - M$, such that the collective choice rule f does not satisfy the equilibrium condition on the preferences of the set N.

We may consider three equilibrium conditions on a collective choice rule. Suppose the collective choice rule gives rise to a weak preference relation R. If, whenever the society weakly prefers an alternative x to an alternative y,

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and an alternative y to an alternative z, then it prefers x to z; say that the collective choice rule is transitive. If this transitivity holds for the strict preference relation P, then say the collective choice rule is quasi-transitive. If it is the case that x_i is strictly preferred to x_{i+1}, for $i = 1,\ldots r$, say, and as a consequence x_{r+1} is not strictly preferred to x_1, then say the collective choice rule is acyclic. Obviously, transitivity implies quasi-transitivity, implies acyclicity. The converse of these conditions are called intransitivity, quasi-intransitivity, and cyclicity.

Let us concern ourselves with the collective choice rule of majority decision making. Sen [10, p. 186] has shown that a particular exclusion principle called value restriction is necessary, in the above sense, and sufficient for majority rule to be quasi-transitive. Suppose, however, that three individuals labeled i, j, k have preferences over three alternatives a, b, c such that i prefers a to b to c, j prefers b to c to a, and k prefers c to a to b. Then say the three individual display cyclic preferences on a, b, c. If it is the case that three individuals exhibit cyclic preferences, then value restriction fails. In fact, all the exclusion principles which guarantee transitivity of the strict preference relation associated with majority rule will fail. Thus triple cyclicity *may* be compatible with intransitivity or cyclicity in the strict preference relation of majority rule.

Kramer [5] turned attention from the usual Arrovian binary choice setting, wherein individual preferences are represented by relations on a discrete set of alternatives W, to the situation where preferences are represented by convex differentiable utility functions defined on a Euclidean space of alternatives. Using the convexity assumption, he was able to develop a pointwise condition on a triple of individual utility functions sufficient to guarantee that these individuals displayed cyclic preferences. Since this pointwise condition is likely to be satisfied for some triple of individuals for at least one point in W, his result cast doubt on the reasonableness of the exclusion principles, at least in the continuous setting.

In this paper we extend Kramer's result by showing that the convexity assumption is not necessary. In fact it is possible to show that the result holds when W has no Euclidean metric properties at all, but has instead the structure of a differentiable manifold. All that is required is the linear structure of the tangent space.

Now Kramer's condition, while it is sufficient to guarantee the cyclicity of preferences of a triple of individuals, is not sufficient to guarantee the cyclicity of preferences of the whole collectivity under a general collective choice rule. Here we present a natural generalization of Kramer's condition which does guarantee that the particular decision rule in question is cyclic.

Arrow's theorem, while generally taken to refer to voting games, is of course relevant to all games of collective choice. The nature of the pointwise condition developed here suggests that it will be satisfied for at least one point for almost all preference patterns in almost all games; thus cyclicity will occur in almost any dynamical game. This suggestion is made more precise in [8].

1. THE TRIPLE-RANK CONDITION AND CYCLICITY

Let $N = \{1,\ldots, n\}$ be a collectivity of individuals. In this section we shall assume that the set of alternatives has the structure of Euclidean m space, with the usual topology. Each individual i in N has a well-defined, differentiable utility function $u_i : W \to R$. The strict preference relation P_i of individual i may be defined by

$$xP_i y \Leftrightarrow u_i(x) > u_i(y) \qquad \text{for } x, y \in W.$$

If M is a subset, or coalition, inside N, then define the M-preference relation P_M by

$$xP_M y \Leftrightarrow xP_i y \qquad \text{for all } i \in M, \text{ for } x, y \in W.$$

Suppose $c : I \to W$ is a smooth path in W, where I is a closed interval in the positive reals. Then we may write

$$c(t) = (c_1(t), c_2(t),\ldots, c_m(t)) \qquad \text{for } t \in I.$$

At a point $\alpha = c(t)$ in W, we may write, for the differential of c,

$$c'(t) = \left(\frac{dc_1}{dt}(t),\ldots, \frac{dc_m}{dt}(t)\right).$$

Such a vector $c'(t)$ will be called a tangent vector at α in W. The set of such tangent vectors at α is the tangent space $T_\alpha W$ at α. However, we may regard $T_\alpha W$ as R^m.

Since $u_i : W \to R$ is differentiable, we may write

$$Du_i(\alpha) = \left(\frac{\partial u_i}{\partial \alpha_1}(\alpha),\ldots, \frac{\partial u_i}{\partial \alpha_m}(\alpha)\right)$$

for the direction gradient of u_i at the point $\alpha = (\alpha_1,\ldots, \alpha_m)$. If $c'(t)$ is a tangent vector in $T_\alpha W$ (i.e., R^m), where $\alpha = c(t)$, then write

$$Du_i(\alpha)(c'(t)) = \sum_{j=1}^{m} \frac{\partial u_i}{\partial \alpha_j}(\alpha) \frac{dc_j}{dt}(t).$$

Since $u_i : W \to R$ is differentiable, we may regard $Du_i : W \to L(R^m, R)$ as a continuous map from W to $L(R^m, R)$, the vector space of linear maps from R^m to R.

Suppose $c : I \to W$ is a smooth path in W such that

$$Du_i(c(t))(c'(t)) > 0 \qquad \text{for all } t \in I,$$

then it is clear that

$$u_i(c(t_2)) > u_i(c(t_1)) \qquad \text{for any } t_2 > t_1 .$$

Thus, the path c may be regarded as an optimizing path for individual i, since his utility increases incrementally down the path. Similarly, if c is an optimizing path for every individual in a coalition M, then we may regard c as an optimizing path for this coalition.

Let M be a coalition for size r. Define

$$Du^M : W \to L(R^m, R^r)$$

to be the continuous map given by

$$Du^M(\alpha)(v) = \begin{pmatrix} Du_{i_1}(\alpha)(v) \\ Du_{i_r}(\alpha)(v) \end{pmatrix}_{i_j \in M} .$$

Since $Du^M(\alpha)$ is linear, the image, $\operatorname{Im} Du^M(\alpha)$, is a vector subspace of R^r; the kernel, $\operatorname{Ker} Du^M(\alpha)$, is a vector subspace of R^m; and

$$\dim \operatorname{Im} Du^M(\alpha) + \dim \operatorname{Ker} Du^M(\alpha) = m.$$

Let V be a subspace of R^r. Write $V^\perp$ for the perpendicular subspace

$$V^\perp = \{x \in R^r : x \cdot v = 0 \text{ for all } v \in V\}.$$

Now let us consider a set of three individuals labeled 1, 2, 3 for convenience, and let $M = \{1, 2, 3\}$. Then

$$Du^M : W \to L(R^m, R^3)$$

is continuous.

Definition 1.1. Say Du^M satisfies the triple-rank condition (TR) at a point α in W iff there exists a strictly positive vector v in R^3 such that $v^\perp$ is a subspace of $\operatorname{Im} Du^M(\alpha)$.

DEFINITION 1.2. Say that three individuals $\{1, 2, 3\}$ exhibit cyclic preferences if there exist points a, b, c in W such that

$$\begin{aligned} u_1(a) &> u_1(b) > u_1(c), \\ u_2(b) &> u_2(c) > u_2(a), \\ u_3(c) &> u_3(a) > u_3(b). \end{aligned} \qquad \blacksquare$$

We now seek to show that if the triple-rank condition is satisfied by Du^M at a point α, then these three individuals exhibit cyclic preferences. We must now introduce some notation.

(i) *Conic Mappings*

A set C of vectors in R^m is a cone iff $x \in C$ implies $\lambda x \in C$, where λ is a strictly positive number. A cone C is convex iff $x, y \in C$ implies $\lambda x + (1 - \lambda) y \in C$ for any λ, with $0 \leq \lambda \leq 1$. A conic mapping $C : W \rightarrow R^m$ is a point set mapping where $C(\alpha)$ is a cone, possibly empty, in R^m.

Remember that we defined $T_\alpha W$ to be the space of tangent vectors at a point α in W. Now define

$$TW = \bigcup_{\alpha \in W} T_\alpha W$$

to be the tangent bundle of tangent vectors to W. A vector field is a smooth function $X : W \rightarrow TW$ where $X(\alpha)$ is a vector in the tangent space $T_\alpha W$. If a vector field X is defined on W, then one can obtain a path $c : I \rightarrow W$ through any point $\alpha \in W$, by taking

$$X(c(t)) = c'(t) \qquad \text{for } t \in I.$$

Say a conic mapping $C : W \rightarrow TW$ is continuous if there is a vector field $X : W \rightarrow TW$ such that $X(\alpha) \in C(\alpha)$ for all $\alpha \in W$. Note that if $C(\alpha) = \varnothing$ for some point α, then we will take α to be a rest point for the vector field, i.e., $X(\alpha) = 0$.

(ii) *Sigma Vectors*

A sigma vector $\sigma = (\sigma_1, \ldots, \sigma_r)$ is a vector where each component σ_i is $+1$, -1, or 0.

Write $R_\sigma{}^r$ for the quadrant of R^r given by

$$R_\sigma{}^r = \left\{ \begin{array}{rl} (x_1, \ldots, x_r): & x_i > 0 \text{ if } \sigma_i = +1 \\ & x_i = 0 \text{ if } \sigma_i = 0 \\ & x_i < 0 \text{ if } \sigma_i = -1 \end{array} \right\}.$$

Where $r = 3$, fix the following notation for the rest of this section:

$$\begin{aligned}\sigma_1 &= (+1, -1, 0),\\ \sigma_2 &= (0, +1, -1),\\ \sigma_3 &= (-1, 0, +1). \quad \blacksquare\end{aligned}$$

Define three point set mappings $C^i : W \dashrightarrow TW$, $i = 1, 2, 3$, by

$$C^i(\alpha) = \{w \in T_\alpha W \colon Du^M(\alpha)(w) \subset R^3_{\sigma_i}\}.$$

Thus, for example, if $i = 1$, then a vector w in $C^1(\alpha)$ has the property that

$$\begin{aligned}Du_1(\alpha)(w) &> 0,\\ Du_2(\alpha)(w) &< 0,\\ Du_3(\alpha)(w) &= 0,\end{aligned}$$

and similarly for $i = 2$ or 3.

We now prove the main theorem of this section.

THEOREM 1. *If the triple-rank condition on* $\{Du_1, Du_2, Du_3\}$ *is satisfied at a point* α, *then there is a neighborhood* $U(\alpha)$ *of* α, *and three points* a, b, c *in* $U(\alpha)$ *such that the three individuals exhibit cyclic preferences on* $\{a, b, c\}$.

Proof. Let $v = (v_1, v_2, v_3)$ be the strictly positive vector of Definition 1.1. Then

(a) $p_1 = (v_2, -v_1, 0) \in \operatorname{Im} Du^M(\alpha) \cap R^3_{\sigma_1}$,

(b) $p_2 = (0, v_3, -v_2) \in \operatorname{Im} Du^M(\alpha) \cap R^3_{\sigma_2}$,

(c) $P_3 = (-v_3, 0, v_1) \in \operatorname{Im} Du^M(\alpha) \cap R^3_{\sigma_3}$.

Consider p_1. There exists some w_1 in $T_\alpha W$ such that

$$Du^M(\alpha)(w_1) = p_1 \in R^3_{\sigma_1}.$$

Thus at this point α, $C^1(\alpha)$ is nonempty. If the triple-rank condition holds at the point α, then $C^1(\alpha)$, $C^2(\alpha)$, and $C^3(\alpha)$ are all nonempty. Now by Corollary 2.5 of the next section, if the triple-rank condition is satisfied at a point α, then there is a neighborhood $U(\alpha)$ of α, where it is satisfied.

Consider now the set of points

$$W_1 = \{x \in W : u_3(x) = u_3(\alpha)).$$

Since u_3 is differentiable, W_1 is a smooth submanifold or hypersurface of W, and contains the point α. Consider the conic mapping restricted to W_1:

$$C^1 : W_1 \to TW_1 .$$

Suppose that $\alpha' \in W_1$. Then a vector w belongs to $C^1(\alpha')$ iff

$$\begin{aligned} Du_1(\alpha')(w) &> 0, \\ Du_2(\alpha')(w) &< 0, \\ Du_3(\alpha')(w) &= 0. \end{aligned}$$

Now by Smale [12], C^1 will be continuous on W_1. If we restrict C^1 further to the neighborhood $U(\alpha)$ of α, then since the triple rank condition is satisfied everywhere in $U(\alpha)$, it is clear that C^1 will be nonempty on $U(\alpha)$. Thus there is a nonzero vector field

$$X_1 : W_1 \cap U(\alpha) \to T(W_1 \cap U(\alpha)),$$

with

$$X_1(\alpha') \in C^1(\alpha') \qquad \text{for all } \alpha' \in W_1 \cap U(\alpha).$$

This means that there is a path $c_1 : I_1 \to W_1 \cap U(\alpha)$ starting from the point α, such that

$$c_1'(t) = X_1(c_1(t)) \in C^1(c_1(t)),$$

and

$$c_1'(t) \neq 0$$

for all $t \in I_1$,

Thus we have a path $c_1 : I_1 \to W_1 \cap U(\alpha)$ with the property that $Du^M(c_1(t))(c_1'(t)) \in R^3_{\sigma_1}$, for all $t \in I_1$. In other words,

$$\begin{aligned} Du_1(c_1(t))(c_1'(t)) &> 0, \\ Du_2(c_1(t))(c_1'(t)) &< 0, \\ Du_3(c_1(t))(c_1'(t)) &= 0. \end{aligned}$$

For some $t_1 \in I_1$ strictly positive let $a = c_1(t_1)$. Since utility for individual 1 increases, utility for 2 decreases, and utility for 3 is constant, along the path c_1 we see that

$$\begin{aligned} u_1(a) &> u_1(\alpha), \\ u_2(a) &< u_2(\alpha), \\ u_3(a) &= u_3(\alpha). \end{aligned}$$

Repeating the procedure on $W_2 = \{x \in W : u_1(x) = u_1(\alpha)\}$, we obtain a path $c_2 : I_2 \to W_2 \cap U(\alpha)$ with $Du^M(c_2(t))(c_2'(t)) \in R^3_{\sigma_2}$ for all $t \in I_2$. Letting $b = c_2(t_2)$ for some $t_2 \in I_2$, $t_2 \neq 0$, we obtain

$$\begin{aligned} u_1(b) &= u_1(\alpha), \\ u_2(b) &> u_2(\alpha), \\ u_3(b) &< u_3(\alpha). \end{aligned}$$

Again taking $W_3 = \{x \in W : u_2(x) = u_2(\alpha))$, we obtain $c_3 : I_3 \to W_3 \cap U(\alpha)$ and a point $c = c_3(t_3)$ for some $t_3 \in I_3$, $t_3 \neq 0$, such that

$$\begin{aligned} u_1(c) &< u_1(\alpha), \\ u_2(c) &= u_2(\alpha), \\ u_3(c) &> u_3(\alpha). \end{aligned}$$

Using these results, we obtain the cyclic preferences

$$\begin{aligned} u_1(a) &> u_1(\alpha) = u_1(b) > u_1(c), \\ u_2(b) &> u_2(\alpha) = u_2(c) > u_2(a), \\ u_3(c) &> u_3(\alpha) = u_3(a) > u_3(b) \end{aligned}$$

on the three alternatives $\{a, b, c\}$ all lying in $U(\alpha)$. ∎

In Fig. 1 we present the typical simple situation. I_1, I_2, and I_3 are the indifference curves of individuals 1, 2, 3 through the point α. The curves are drawn as though the utility functions were convex. As an exercise the reader

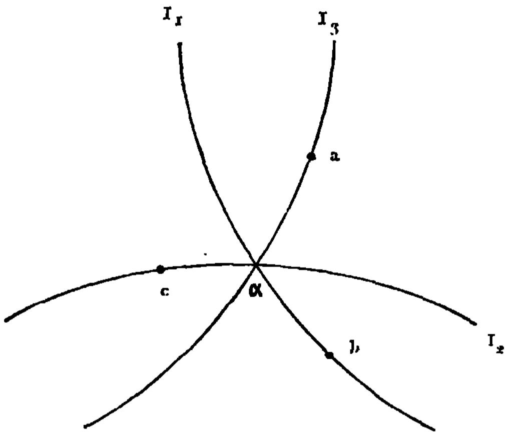

FIGURE 1

may like to draw indifference curves representing nonconvex utility functions and find the appropriate neighborhood $U(\alpha)$ of α.

The result, that there is a neighborhood $U(\alpha)$ of α where the cones C^1, C^2, and C^3 are nonempty, is crucial for the proof of this theorem. The next section is devoted to the proof of this result. We seek now, however, to relate the above theorem to Kramer's result.

Suppose $A_r = \{a_1, \dots, a_r\}$ is a set of r vectors in R^m. A semipositive combination of A_r is any vector of the form

$$a = \sum_{j=1}^{r} \lambda_j a_j ,$$

where each λ_j is nonnegative, and not all are zero. Write $\mathrm{SP}(A_r)$ for the set of semipositive combinations of A_r. If it is the case that $0 \in \mathrm{SP}(A_r)$, then say the vectors in A_r are semipositively dependent. Similarly, write

$$\mathrm{P}(A_r) := \left\{ a \in R^m : a = \sum_{j=1}^{r} \lambda_j a_j : \lambda_j > 0 \right\}$$

for the strictly positive span of A_r. If $0 \in \mathrm{P}(A_r)$, then say the vectors in A_r are positively dependent.

DEFINITION 1.3. Regard $Du_1(\alpha)$, $Du_2(\alpha)$, $Du_3(\alpha)$ as vectors in R^m. Say this triple satisfies Kramer's condition (K) at a point α in W iff each vector is nonzero, no one can be expressed as a positive combination of the other two, and no two are linearly dependent. Condition K is obviously equivalent to the following: Either (i) $\{Du_1(\alpha), Du_2(\alpha), Du_3(\alpha)\}$ are linearly independent, or (ii) $\{Du_1(\alpha), Du_2(\alpha), Du_3(\alpha)\}$ are positively dependent. ▌

We now seek to show that Kramer's condition and the triple-rank condition are equivalent. Let $\{a_1, \dots, a_r\}$ be a set of r vectors in R^m. Regard

$$A = \begin{bmatrix} a_1 \\ a_r \end{bmatrix}_{1,\dots,r}$$

as a linear map $A : R^m \to R^r$. Suppose $\lambda \in R^r$. We may regard $\lambda \cdot A$ either as a vector in R^m or as a linear map $R^m \to R$ by

$$(\lambda \cdot A)(x) := \lambda \cdot (A(x)),$$

where $x \in R^m$.

LEMMA 1.1. *Suppose the vectors* $\{a_1, \dots, a_r\}$ *satisfy a relation*

$$\sum_{j=1}^{r} \lambda_i a_i = 0.$$

Then $(\mathrm{Im}\ A)^{\perp}$ *contains the subspace of* R^r *spanned by the vector*

$$\lambda = (\lambda_1, \ldots, \lambda_r).$$

Proof.

$$\begin{aligned}(\mathrm{Im}\ A)^{\perp} &= \{w \in R^r : w \cdot x = 0 \text{ for all } x \in \mathrm{Im}\ A\}\\ &= \{w \in R^r \text{ such that } w \cdot (A(y)) = 0 \text{ for all } y \in R^m\}\\ &= \{w \in R^r : (w \cdot A)(y) = 0\ \forall y \in R^m\}\\ &\supset \{w \in R^r : (w \cdot A) = 0\}.\end{aligned}$$

But the relation

$$\sum_{i=1}^{r} \lambda_i a_i = 0$$

is equivalent to $\lambda \cdot A = 0$. Thus $\lambda \in (\mathrm{Im}\ A)^{\perp}$. ▮

LEMMA 1.2. *A necessary and sufficient condition for* $\{Du_1(\alpha), Du_2(\alpha), Du_3(\alpha)\}$ *to satisfy Kramer's condition is that* Du^M *satisfy the triple-rank condition.*

Proof. (i) If the triple is linearly independent, then

$$\mathrm{Im}\ Du^M(\alpha) = R^3.$$

Hence, the triple-rank condition is satisfied.

(ii) Suppose

$$\sum_{i=1}^{3} \lambda_i\ Du_i(\alpha) = 0, \qquad \text{with} \quad \lambda_1, \lambda_2, \lambda_3 > 0.$$

By Lemma 1.1,

$$(\lambda_1, \lambda_2, \lambda_3) \in [\mathrm{Im}\ Du^M(\alpha)]^{\perp}.$$

In fact, $\lambda^{\perp} = (\lambda_1, \lambda_2, \lambda_3)^{\perp} = \mathrm{Im}\ Du^M(\alpha)$. Thus the triple-rank condition is satisfied.

(iii) Suppose $\{Du_i(\alpha) : i = 1, 2, 3\}$ are linearly dependent, but are not positively dependent. Since the triple-rank condition cannot be true if $\dim(\mathrm{Im}\ Du^M(\alpha)) = 1$, we need consider only those cases where $\mathrm{Im}\ Du^M(\alpha) = \lambda^{\perp}$, $\lambda \in R^3$. Now the gradient vectors are not positively dependent, so

$$\lambda \in R^3 - R_{\sigma}^{3},$$

where $\sigma = (+1, +1, +1)$ by Lemma 1.1. But for the triple-rank condition to be satisfied we require $\lambda \in R_\sigma^3$, which is a contradiction. ▌

COROLLARY 1.3 (Kramer's Theorem). *If at a point* $\alpha \in W$, *there exist three individuals whose direction gradients* $\{Du_1(\alpha), Du_2(\alpha), Du_3(\alpha)\}$ *are such that no one can be expressed as a positive linear combination of the other two, and no two of which are linearly dependent, then the individuals exhibit cyclic preferences.* ▌

COROLLARY 1.4. *For any exclusion principle to hold it is necessary that at any point* α *in* W, *either*

(i) *there be a pair of individuals* $\{1, 2\}$, *say, such that*

$$Du_i(\alpha) = \lambda_1 Du_1(\alpha) + \lambda_2 Du_2(\alpha) \qquad \text{for all } i \in N : \lambda_1, \lambda_2 > 0,$$

or

(ii) *there be one individual* 1, *say, such that*

$$Du_i(\alpha) = \lambda Du_1(\alpha) \qquad \text{for all } i \in N, \ \lambda \neq 0. \quad ▌$$

Since one would imagine that in general there would be at least one point in W and some triple of individuals whose direction gradients satisfied the Kramer condition, all exclusion principles can be expected to fail. This is not to say that the collective choice rule is necessarily cyclic. Below we give an example of a four-person game on a two-dimensional space W, where individuals $\{1, 2, 3\}$ and $\{2, 3, 4\}$ both have cyclic preferences near a point α, but no cycle of the voting game with quota 3 exists near the point.

2. THE NULL DUAL CONDITION

In the previous section we defined a differentiable utility function in terms of a continuous function

$$Du_i : W \to R^m.$$

Since the length of $Du_i(\alpha)$ is irrelevant, let us factor it out as follows. If a_1 and a_2 are two vectors in R^m, say $a_1 \sim a_2$ iff $a_1 = \lambda a_2$ for some $\lambda > 0$. Write S^m for R^m under this equivalence relation. On S^m define the natural topology by the metric

$$d([x], [y]) = \left[2 - \frac{2(x \cdot y)}{\| x \| \| y \|}\right]^{1/2},$$

where x and y are elements in the classes $[x]$, $[y]$. An element in S^m is called a cone. Say that an individual i has an ordinal differentiable utility function if there is a function $Du_i : W \to S^m$, which is continuous with respect to the above topology on S^m and represents i's preferences. For the moment we can avoid the problem of bliss points where $Du_i(\alpha) = 0$.

Assume that each individual i in N has such an ordinal differentiable utility function. We can define a dual mapping $*$: $S^m \to S^m$ by

$$[a]^* = \{[x] \in S^m : a \cdot x > 0 \text{ for all } x \in [x] \text{ and all } a \in [a]\},$$

which obviously extends to sets. For each individual i, define his preference cone at a point α to be

$$C_i(\alpha) = Du_i(\alpha)^*.$$

We may regard this preference cone as the set of vectors

$$\{v \in R^m : Du_i(\alpha)(v) > 0\}.$$

If M is a coalition, define its preference cone at α to be

$$C_M(\alpha) = \bigcap_{i \in M} C_i(\alpha).$$

If $[a]$, $[b]$ are two-point cones in S^m, define their conic addition to be

$$[a] + [b] = [\lambda_1 a + \lambda_2 b],$$

where $\lambda_1 a + \lambda_2 b$ is a semipositive combination of a and b. A cone C in S^m is finitely generated if any point cone $[c]$ in C can be written in the form

$$[c] = [a_1] + \ldots + [a_r]$$

for some fixed and finite set $\{[a_i] : i = 1, \ldots, r\}$. We shall use the notation

$$C = \sum_{i=1}^{r} [a_i].$$

We give here a few results without proof on the dual mapping, which are adapted from [3].

LEMMA 2.1. *(i) If* $C_1 \subset C_2$, *then* $C_2^* \subset C_1^*$.

(ii) $(C_1 \cup C_2)^* = C_1^* \cap C_2^*$.

(iii) *If C is convex then $C = C^{**}$. Note that if C is finitely generated then it is convex.*

(iv) *If C is convex then exactly one of the following is true:*

(a) $C^* \neq \varnothing$,

(b) $[0] \in C$. *In other words, the generators of C are semipositively dependent.*

(v) *If* $C = \sum_{i=1}^{r} [a_i]$, *then* $C^* = \bigcap_{i=1}^{r} [a_i]^*$. ▮

Let us define the preference co-cone of a coalition M to be

$$Du_M(\alpha) = \sum_{i \in M} Du_i(\alpha),$$

where this is a conic addition. Then

$$\begin{aligned} Du_M(\alpha)^* &= \Big[\sum_{i \in M} Du_i(\alpha)\Big]^* \\ &= \bigcap_{i \in M} Du_i(\alpha)^* \qquad \text{by Lemma 2.1(v),} \\ &= \bigcap_{i \in M} C_i(\alpha) \\ &= C_M(\alpha). \end{aligned}$$

By Lemma 2.1(iv), a necessary and sufficient condition for $Du_M(\alpha)^* = C_M(\alpha)$ to be empty is that $[0] \in Du_M(\alpha)$, i.e., that $\{Du_i(\alpha) : i \in M\}$ be semipositively dependent. If $C_M(\alpha) = \varnothing$, then define $C_M(\alpha)^*$ to be S^m. If $C_M(\alpha) \neq \varnothing$, then by Lemma 2.1(iii),

$$Du_M(\alpha)^{**} = C_M(\alpha)^* = Du_M(\alpha).$$

We have thus established a nice duality between the cone $C_M(\alpha)$ and its co-cone $Du_M(\alpha)$.

DEFINITION 2.1. Let ω be some set of coalitions. Say the null dual condition is satisfied at a point α in W, with respect to ω, iff the set of preference cones

$$C^{\omega}(\alpha) = \bigcup_{M \in \omega} C_M(\alpha)$$

has empty dual, i.e.,

$$C^{\omega}(\alpha)^* = \varnothing.$$

LEMMA 2.2. *The null dual condition at a point* α *is satisfied with respect to* ω, *iff*

$$\bigcap_{M\in\omega} Du_M(\alpha) = \emptyset,$$

where the intersection is over all $Du_M(\alpha)$, *for* M *in* ω, *with*

$$C_M(\alpha) \neq \emptyset.$$

Proof.

$$\begin{aligned} C^{\omega}(\alpha)^* &= \bigcap_{\omega} C_M(\alpha)^*, && \text{by Lemma 2.1(ii),} \\ &= \bigcap_{\omega} Du_M(\alpha)^{**} \\ &= \bigcap_{\omega} Du_M(\alpha) && \text{by Lemma 2.1(iii).} \quad \blacksquare \end{aligned}$$

DEFINITION 2.2. A conic mapping $C : W \dashrightarrow S^m$ is upper semicontinuous (usc) at a point α in W if, for each open set G containing $C(\alpha)$, there is a neighborhood $U(\alpha)$ of α in W such that

$$\alpha' \in U(\alpha) \Rightarrow C(\alpha') \in G.$$

The conic mapping is usc in W if C is usc at all α in W, and $C(\alpha)$ is compact for all α in W.

LEMMA 2.3. *For any coalition* M, $Du_M : W \to S^m$ *is upper semicontinuous.*

Proof. $Du_M(\alpha)$ is compact for all α in W, since it is by definition closed. Let $G' \supset Du_M(\alpha)$, with G' open. We may in fact take an open convex neighborhood G of $Du_M(\alpha)$ in G'. By the continuity of each $Du_i : W \to S^m$, there is some neighborhood $U^i(\alpha)$ such that

$$\alpha' \in U^i(\alpha) \Rightarrow Du_i(\alpha') \in G.$$

Thus, if

$$U(\alpha) = \bigcap_{i\in M} U^i(\alpha),$$

$$\alpha' \in U(\alpha) \Rightarrow Du_M(\alpha') \subset G. \quad \blacksquare$$

See [7] for further details of this proof.

THEOREM 2. *If the null dual condition is true for* ω *at a point* α, *then there* *is a neighborhood $U(\alpha)$ of α in W such that the null dual condition is true for ω at all points in $U(\alpha)$.*

Proof. We give an outline of the proof here. For further details see [7]. Now

$$C^{\omega}(\alpha)^* = \bigcap_{M\in\omega} Du_M(\alpha) = \varnothing .$$

Assume $C_M(\alpha) \neq \varnothing$, so that $0 \notin Du_M(\alpha)$. Since $Du_M(\alpha)$ is closed, there exist open neighborhoods G_M of each $Du_M(\alpha)$ in $S^m - \{0\}$, such that

$$\bigcap_{M\subset\omega} G_M = \varnothing .$$

Further, these neighborhoods can be chosen to be convex. Since each $Du_M : W \to S^m$ is usc, there exist neighborhoods $U_M(\alpha)$ of α in W such that

$$\alpha' \in U_M(\alpha) \Rightarrow Du_M(\alpha') \subset G_M .$$

Let

$$U(\alpha) = \bigcap_{M\in\omega} U_M(\alpha).$$

Thus

$$\alpha' \in U(\alpha) \Rightarrow C^{\omega}(\alpha')^* = \bigcap_{\omega} Du_M(\alpha') \subset \bigcap_{\omega} G_M = \varnothing . \quad \blacksquare$$

Note, in fact, that.

$$G_M{}^* \subset Du_M(\alpha')^* = C_M(\alpha')$$

and

$$\begin{aligned} \Big(\bigcup_{\omega} G_M{}^*\Big)^* &= \bigcap_{\omega} G_M^{**} \\ &= \bigcap_{\omega} G_M , \qquad \text{since each is convex,} \\ &= \varnothing . \end{aligned}$$

Thus the null dual property can be maintained in a neighborhood of α in W by a fixed set of cones $\{G_M{}^* : G_M{}^* \subset C_M(\alpha),\ M \in \omega\}$.

We now relate the null dual condition to Kramer's condition.

LEMMA 2.4. *Let $\{1, 2, 3\}$ be three individuals, and let*

$$\omega = \{1, 2\} \cup \{2, 3\} \cup \{1, 3\}.$$

Then $C^{\omega}(\alpha)$ satisfies the null dual condition iff Kramer's condition is satisfied.

Proof. If $p \in Du_{12}(\alpha)$, say, then we may write

$$p = \lambda_1 Du_1(\alpha) + \lambda_2 Du_2(\alpha),$$

a semipositive combination. By Lemma 2.2 $p \in C^{\omega}(\alpha)^*$ when

$$p \in Du_{12}(\alpha) \cap Du_{13}(\alpha) \cap Du_{23}(\alpha).$$

Thus for $C^{\omega}(\alpha)^* \neq \varnothing$ we require solutions to the three semipositive equations

$$\begin{aligned} p &= \lambda_1 Du_1(\alpha) + \lambda_2 Du_2(\alpha) && \text{(i)} \\ &= \lambda_3 Du_1(\alpha) + \lambda_4 Du_3(\alpha) && \text{(ii)} \\ &= \lambda_5 Du_2(\alpha) + \lambda_6 Du_3(\alpha). && \text{(iii)} \end{aligned}$$

Suppose Kramer's condition is satisfied at α. Then the only solution to Eqs. (i) and (ii) is $p = \lambda Du_1(\alpha)$ for some $\lambda > 0$.

The existence of a semipositive solution

$$Du_1(\alpha) = \lambda_5 Du_2(\alpha) + \lambda_6 Du_3(\alpha)$$

is impossible, when the direction gradients are linearly independent, or strictly positively dependent. Consider the remaining situations. Suppose that there is a positive solution to

$$Du_1(\alpha) + \lambda Du_2(\alpha) = 0.$$

By Lemma 2.1(iv),

$$Du_{12}(\alpha)^* = C_{12}(\alpha) = \varnothing.$$

Thus

$$C^{\omega}(\alpha) = C_{23}(\alpha) \cup C_{13}(\alpha)$$

and

$$\begin{aligned} p = Du_3(\alpha) &\in Du_{13}(\alpha) \cap Du_{23}(\alpha) \\ &= C^{\omega}(\alpha)^*. \end{aligned}$$

Thus $C^{\omega}(\alpha)^* \neq \varnothing$.

Suppose, on the other hand, that

$$Du_1(\alpha) = \lambda_2 Du_2(\alpha) + \lambda_3 Du_3(\alpha),$$

a semipositive combination. Then

$$p = \lambda Du_1(\alpha) \in Du_{12}(\alpha) \cap Du_{13}(\alpha),$$

and also belongs to $Du_{23}(\alpha)$. Thus again, $C^{\omega}(\alpha)^* \neq \varnothing$.

The other cases of nonpositive dependence between the direction gradients can be treated similarly. In each case, there is a solution to the equation $p \in C^{\omega}(\alpha)^*$. ▮

COROLLARY 2.5. *If the triple-rank condition is true at a point* α, *then it is true in some neighborhood of* α.

Proof. By Lemma 1.2, the triple-rank condition and Kramer's condition are equivalent. By Lemma 2.4, Kramer's condition and the null dual condition on the three preference cones of two-person coalitions are equivalent. The result follows by Theorem 2. ▮

We can interpret the null dual condition in another way. A cone C belongs to an open half space iff there is some point cone v in S^m such that $v \cdot c > 0$ for all c in C. If $C^{\omega}(\alpha)^* \neq \emptyset$, then the set of cones $C^{\omega}(\alpha)$ belongs to some open half-space. If ω is a set of winning coalitions, say, then any optimizing behavior by the coalitions will be directed toward this half space. Thus $C^{\omega}(\alpha)^*$ can be interpreted as a direction gradient for the social preference function. To illustrate this, in Fig. 2 we represent three typical direction gradients satisfying positive dependence. From Fig. 3 it is clear that the associated two-person cones do not all lie in a half-space. Alternatively, in Fig. 4 we see that the two-person co-cones do not intersect.

This suggests a conjecture which will be proved in detail in [7].

Conjecture. Let ω be a set of winning coalitions of a simple quota game. If $C^{\omega}(\alpha)^* = \emptyset$ at a point α in W, there is a neighborhood $U(\alpha)$, and a cycle of the voting game inside $U(\alpha)$.

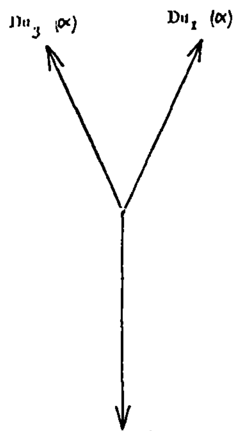

FIGURE 2

DEFINITION 3.2. An m-dimensional differentiable manifold W is a topological manifold with a differentiable structure D. Here D consists of a collection of coordinate neighborhoods (U, h) on W, where $h\colon U \to R^m$ is a homeomorphism, satisfying

(i) the coordinate neighborhoods cover W,

(ii) if (U_1, h_1) and (U_2, h_2) are coordinate neighborhoods with $h_i : U_i \to R^m$, then

$$h_1 h_2^{-1}\colon h_2(U_1 \cap U_2) \subset R^m \to R^m$$

is differentiable.

(iii) the collection is maximal, in the sense that if a coordinate neighborhood is adjoined to D, then property (ii) fails.

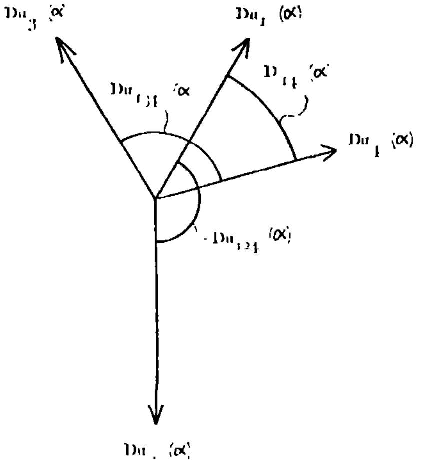

FIGURE 6

Essentially, a differentiable manifold is a topological space which is smoothly and locally Euclidean.

Consider a map $c : I \to W$, where I is an interval in R. Suppose for any x in the image of c, and any coordinate neighborhood (U, h) of x, the map $hc : I \to R^m$ is differentiable; then call c a smooth curve in W. A curve at α is a smooth curve $c : I \to W$ such that $c(0) = \alpha$. Two curves c_1, c_2 at α are tangent if $h \circ c_1$, and $h \circ c_2$ are tangent for any coordinate neighborhood (U, h) of α. In other words,

$$(d/dt)(h \circ c_1) = (d/dt)(h \circ c_2) \qquad \text{at } \alpha.$$

The equivalence class of curves tangent to one another at α is written $[c](\alpha)$, and called a tangent vector at α. The set of tangent vectors at a point α is written $T_\alpha W$. Essentially, a tangent vector at α may be represented as an m vector

$$v = ((d/dt)(c_1)(t), \ldots, (d/dt)(c_m)(t))$$

under a coordinate system (U, h), where $h : U \to R^m$ is given by

$$h(\alpha) = (h_1(\alpha), \ldots, h_m(\alpha)),$$

and $c_i : I \to R$ is given by

$$c_i(t) = h_i(c(t)).$$

Thus the form of the tangent vector at a point is dependent on the particular coordinate neighborhood chosen.

Suppose (U, h) and (V, k) are two coordinate systems about α, and v_1, v_2 are the corresponding m vector representations. Then the transformation from v_1 to v_2 is given by

$$v_2 = D(kh^{-1})(\alpha) \cdot v_1,$$

where $D(kh^{-1})(\alpha) : R^m \to R^m$ is the Jacobian of the map $h^{-1}k$, at α. Let $TW = \bigcup_\alpha T_\alpha W$ be called the tangent bundle of W, and $\pi : TW \to W$, the map which carries a tangent vector at α to the point α, be called the projection mapping.

If (U, h) is a coordinate neighborhood of α, write v for the tangent vector, and $v(h)$ the representation under the coordinate system, as above. Define

$$\pi(h) : \pi^{-1}(U) \to U \times R^m \to R^m \times R^m$$

by

$$[\pi(h)](v) = h \circ \pi(v) \times v(h).$$

Thus $(\pi^{-1}(U), \pi(h))$ is a coordinate neighborhood of $\pi^{-1}(\alpha)$, and in this way TW can be given the structure of a $2m$-dimensional differentiable manifold.

Suppose (U, h) is a coordinate neighborhood of a point α. Then

$$\pi(h) : \pi^{-1}(U) \to U \times R^m$$

is a diffeomorphism, and a linear isomorphism of the tangent space $T_\alpha W$

onto $\{\alpha\} \times R^m$. If (V, k) is a second coordinate neighborhood of α, then there is a linear diffeomorphism $g : R^m \to R^m$ so that

$$\pi(k) = (1 \times g) \circ \pi(h) \qquad \text{on } \pi^{-1}(U \cap V).$$

What this means is that locally the tangent space has a natural linear structure. In a neighborhood of a point we may choose a coordinate system, and represent tangent vectors, and paths, with reference to this coordinate system.

There may be no linear structure on the manifold W itself. Suppose $u : W \to R$ is a map. If W has a linear structure say that u is convex if $u^{-1}[t, \infty)$ is convex for all $t \in R$. With no linear structure on W, we cannot speak of the convexity of a (utility) function on W.

However. if $C : W \to TW$ is a preference cone, as introduced in Section 2, with $C(\alpha) \subset T_\alpha W$ for all $\alpha \in W$, then the linear structure on $T_\alpha W$ allows us to speak of the convexity of C.

Suppose $u : W \to R$ is smooth, that is, differentiable with respect to a coordinate system. Let $[c](\alpha)$ be a tangent vector at α. Then define

$$Du(\alpha)([c](\alpha)) = [u \circ c](u(\alpha)).$$

This essentially defines a linear map $Du(\alpha) : T_\alpha W \to R$, or $Du : W \to T^*W$, where T^*W is the cotangent bundle of linear maps $TW \to R$. This endows T^*W locally with a natural linear structure. Thus if $D : W \to T^*W$ is a preference co-cone, as in Section 2, one may speak of its convexity.

Since the proofs in Sections 1 and 2 essentially only use the linearity properties of TW and T^*W, rather than any linearity property of W, all the proofs carry over in the natural way to the case when W is a differentiable m manifold.

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Asymmetric Duopoly and the Core*

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It has recently been proposed that an exchange economy with "small" and "large" traders be represented by a measure space whose atoms represent the set of large traders and whose atomless part represents the set of small traders. For such mixed markets, the equivalence of the core and the set of competitive allocations, which holds when the set of atoms is of measure zero, no longer holds necessarily when this set has positive measure. However, if this set consists of identical traders, the equivalence is restored. The asymptotic analogs of these propositions are proved here.

The purpose of this paper is to demonstrate that, in some cases, certain surprising features of "mixed markets" that occur when a nonatomic continuum is used to model the "small" players may also occur when a limiting approach is used.

In [3], Shitovitz has studied the core of "mixed markets" involving a set of "infinitesimal" traders and a set of large traders. Formally, the mixture is represented by a measure space of traders, whose atoms represent the large traders, and whose atomless part represents the set of small traders. Through an example [3, Example 1, p. 486] involving two commodities and a single atom of measure 1, he shows that the equivalence principle (equivalence of the core and the set of competitive allocations), which holds for atomless markets, no longer holds for this mixed market. It turns out that the core is much larger here than the set of competitive allocations.

Shitovitz proves, however (Theorem B), that if the set of atoms consists of at least two traders with identical tastes and resources, the equivalence theorem is restored, whatever the relative size (measure) of the atoms. In particular, if the single atom of Example 1 is replaced by two atoms of measure equal to α and $1 - \alpha$, respectively, the core consists of the sole competitive allocation, irrespective of the value of α in $]0, 1[$.

As Shitovitz himself points out (see his "Concluding Remarks," pp. 497–498), the above conclusions are somewhat disturbing. First, one would not

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expect the core of a duopolistic market to coincide with its competitive solutions; and one would even less expect this result to hold whatever the relative sizes of the duopolists may be. Indeed if, in the above example, the number α is set almost equal to 1, the resulting exchange economy is very close to a monopolistic economy consisting of a single atom of measure 1; however, the large atom in this economy cannot get in the core a better outcome than the competitive one.

Second, Shitovitz's results imply that, in the abstract setup of a measure space of traders with atoms and an atomless part, the core correspondence is not lower hemicontinuous. Consider indeed the sequence of economies E_r obtained by replacing the single atom of Example 1 by two atoms of measures equal to $1/r$ and $1 - (1/r)$, respectively. This sequence of economies converges to the economy E_0 containing a single atom of measure 1. According to Shitovitz's Theorem B, the core of E_r consists, for all r, of the sole competitive allocation. However, for $r = \infty$, the core suddenly enlarges so as to contain many allocations which are not competitive.

Starting from these difficulties, some have questioned the abstract representation of oligopolistic markets via economies containing atoms and an atomless sector. Perhaps the abstract representation itself is responsible for those disturbing results (see [2]). The question is whether the representation of a mixture of small and large traders via finite, possibly large, economies would dispose of these pathologies. We show in this paper that this is *not* the case: The same disturbing phenomena may well occur even if one considers *finite* exchange economies.

To this end, we first define a sequence of finite exchange economies E_r corresponding to Shitovitz's Example 1. It is shown that, for large values of r, this market behaves like a market with one large atom, in which the core is large and tends to be favorable to the atom. Then we consider finite exchange economies in which there are l commodities, one trader of measure $1 - (1/r)$, another trader of the same type but with measure $1/r$, and one type of trader consisting of r traders of measure $1/r$ each. Assuming C^1 utility functions, it is shown that the sequence of cores of these finite exchange economies must converge toward the set of competitive allocations, a result which is the asymptotic analog of Shitovitz's Theorem B.

AN EXAMPLE OF A MONOPOLISTIC MARKET

Let r be any positive integer. Let E_r^M be the exchange economy with two commodities, and involving $r + 1$ traders, one trader of type 1 with initial endowment $w_1^r = (4, 0)$ and r traders of type 2, with initial endowments $w_i^r = (0, 4/r)$. Assume $u_i(x, y) = x^{1/2} + y^{1/2}$, $i = 1,..., r + 1$. If $r = 1$, the core of this economy is the whole segment AB of Fig. 1; the set of competi-

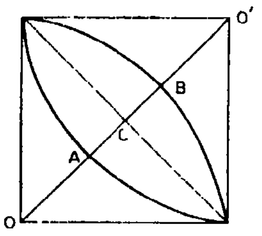

FIGURE 1

tive allocations consists of the sole allocation $x_1 = x_2 = (2, 2)$ (point C in Fig. 1). For large values of r, E_r^M adequately depicts a *finite* exchange economy with one "large" trader having a corner on commodity 1, and facing many "small" traders sharing the ownership of commodity 2, an exchange situation very similar to Shitovitz's Example 1. Clearly the unique competitive allocation of E_r^M is, for all r, $x_1^r = [2(1 - (1/r)), 2(1 - (1/r))]$ and $x_i^r = [2/r, 2/r]$, $i = 2, \dots, r + 1$. Then we have

PROPOSITION 1. *For all r, the set of allocations*

$$X^r = \Big\{(x_1^r, x_2^r, \dots, x_{r+1}^r) \mid x_i^r = \Big(\frac{\gamma_i}{r}, \frac{\gamma_i}{r}\Big); \gamma_i \in [1, 2], i = 2, \dots, r + 1,$$

$$x_1^r = \Big(4 - \frac{1}{r}\sum_{i=2}^{r+1} \gamma_i, 4 - \frac{1}{r}\sum_{i=2}^{r+1} \gamma_i\Big)\Big\}$$

is contained in the core of E_r^M.

Proof. Suppose on the contrary that some allocation $x^r \in X^r$ is not in the core of E_r^m, and let S be the coalition which can improve x^r via the bundles y_i, $i \in S$. We then have

$$\sum_{i\in S} y_i = \sum_{i\in S} w_i^r. \tag{1}$$

Since for all $i \in \{2, \dots, r + 1\}$, $\gamma_i \geqslant 1$, no coalition contained in this set can improve upon x^r; trader 1 is thus a member of S. On the other hand, it is easily seen that x^r is Pareto optimal, so that the coalition of all traders cannot improve upon x^r. Consequently S is a coalition composed of trader 1 and a *proper* subset of $\{2, \dots, r + 1\}$. For all i, x_i^r maximizes u_i on the set $\{y \mid p \cdot y \leqslant p \cdot x_i^r\}$ if p is set equal to $(1, 1)$. Consequently, for all $i \in S$,

$$u_i(y_i) > u_i(x_i^r) \Rightarrow p \cdot y_i > p \cdot x_i^r \Rightarrow p \cdot \sum_{i\in S} y_i > p \cdot \sum_{i\in S} x_i^r. \tag{2}$$

Furthermore for all $i \notin S$, $\gamma_i \leqslant 2$ and $\{1, \dots, r + 1\} \backslash S \subset \{2, \dots, r + 1\} \Rightarrow$

$$p \cdot x_i^r = 2\gamma_i/r \leqslant p \cdot w_i^r = 4/r,$$

which, in turn, implies

$$p \cdot \sum_{i \in \{1,\dots,r+1\} \setminus S} w_i^r \geqslant p \cdot \sum_{i \in \{1,\dots,r+1\} \setminus S} x_i^r .$$

We then have, from (2) and from the fact that x^r is an allocation,

$$\begin{aligned} p \cdot \sum_{i \in S} y_i + p \cdot \sum_{i \in \{1,\dots,r+1\} \setminus S} w_i^r &> p \cdot \sum_{i \in S} x_i^r + p \cdot \sum_{i \in \{1,\dots,r+1\} \setminus S} x_i^r \\ &= p \cdot \sum_{i=1}^{r+1} w_i^r \\ &= p \cdot \sum_{i \in S} w_i^r + p \cdot \sum_{i \in \{1,\dots,r+1\} \setminus S} w_i^r , \end{aligned}$$

an inequality which contradicts (1). Q.E.D.

It follows from Proposition 1 that, for the sequence of "monopolistic" economies $\{E_r^M\}$, we do not have the usual asymptotic property of "purely competitive" economies, i.e., that the core shrinks to the set of competitive allocations when the number of traders increases. In particular, all "equal-treatment" allocations on the segment *CB* of Fig. 1 remain in the core of E_r^M for all r. Accordingly, in this case, the "limit" characterization of the core of E_r^M is consistent with the characterization "at the limit" obtained by Shitovitz.

The next section shows that this consistency also holds for "duopolistic" economies of the kind considered in Shitovitz's Theorem B.

AN ASYMPTOTIC VERSION OF SHITOVITZ'S THEOREM B

Consider the exchange economy E_r defined as follows. There are l commodities, one trader with initial endowment $w_1^r = (1 - (1/r))\, w$, $w \gg 0$, another trader with initial endowment $w_2^r = w/r$, and r traders with initial endowment $w_i^r = w'/r$ each, $i = 3,\dots, r+2$, $w' \gg 0$.[1] Assume that traders 1 and 2 have preference orderings, denoted by $\gtrsim_{1r}$ and $\gtrsim_{2r}$, respectively, derived from a given preference ordering $\gtrsim_1$ by the conditions

$$x \underset{1}{\gtrsim} y \Leftrightarrow x\left(1 - \frac{1}{r}\right) \underset{1r}{\gtrsim} y\left(1 - \frac{1}{r}\right) \Leftrightarrow \frac{x}{r} \underset{2r}{\gtrsim} \frac{y}{r} . \qquad (3)$$

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[1] In this section we consider that there are only two types of traders. The reasoning would easily extend to the case of more types of traders, each consisting of r traders of measure $1/r$ each.

Assume also that all traders in $\{3,\dots, r+2\}$ have the same preference ordering $\succsim_r$ derived from a given preference ordering $\succsim$ by the condition

$$x \succsim y \Leftrightarrow \frac{x}{r} \underset{r}{\succsim} \frac{y}{r}. \tag{4}$$

The following assumptions are made on $\succsim_1$ and $\succsim$: (i) $\succsim_1$ and $\succsim$ are both continuous on the closed positive orthant R_+^l; (ii) $\succsim_1$ and $\succsim$ are both strictly convex and monotone on the (relative) interior of R_+^l;[2] (iii) $\succsim_1$ and $\succsim$ are both representable by C^1 utility functions with nonvanishing gradients; (iv) if x is on the boundary of R_+^l, $x \sim_1 0$ and $x \sim 0$.

Let $N(E_r)$ denote the core of the economy E_r. First, we prove the following:

LEMMA. *If* $(x_1^r, x_2^r, x_3^r,\dots, x_{r+2}^r)$ *is an allocation in* $N(E_r)$, *then there exist two vectors,* x *and* x', *such that* $x_1^r = x(1-(1/r))$, $x_2^r = x/r$, *and* $x_i^r = x'/r$, $i = 3,\dots, r+2$.

Proof. Let the allocation $(x_1^r, x_2^r,\dots, x_{r+2}^r)$ be in $N(E_r)$. First, we show that $x_1^r/(r-1) \succsim_{2r} x_2^r$. Suppose indeed that we would have $x_2^r \succ_{2r} x_1^r/(r-1)$. Then by (3), $(r-1)\,x_2^r \succ_{1r} x_1^r$. Since $\succsim_{1r}$ is strictly convex,

$$y_1^r \underset{\text{def}}{=} \frac{1}{r}(r-1)\,x_2^r + \frac{r-1}{r}\,x_1^r \underset{1r}{\succ} x_1^r.$$

Let i^* be the trader in $\{3,\dots, r+2\}$ such that $x_{i^*}^r \succsim_r x_i^r$ for all $i \in \{3,\dots, r+2\}$. Consider the coalition S defined by $S = \{1 \cup \{3,\dots, r+2\}\backslash i^*\}$ and define y_i for i in S by

$$y_1^r = \frac{r-1}{r}(x_2^r + x_1^r),$$

$$y_i^r = \frac{1}{r}x_{i^*}^r + \left(1-\frac{1}{r}\right)x_i^r, \qquad i = 3,\dots, r+2, \quad i \neq i^*.$$

Clearly, by strict convexity of $\succsim_r$, $y_i^r \succsim_r x_i^r$ for all $i \in \{3,\dots, r+2\}$, $i \neq i^*$. Furthermore,

$$\sum_{i\in S} y_i^r = \frac{r-1}{r}(x_1^r + x_2^r) + \frac{r-1}{r}x_{i^*}^r + \frac{r-1}{r}\sum_{\substack{i=3\\ i\neq i^*}}^{r+2} x_i^r,$$

$$= \frac{r-1}{r}\sum_{i=1}^{r+2} x_i^r$$

$$= \frac{r-1}{r}(w + w') = \sum_{i\in S} w_i^r,$$

---

[2] The monotonicity assumption means: if $x^h \geq y^h$ for all $h = 1,\dots, l$ and $x^k > y^k$ for some k, then $x \succsim_1 y$ (resp. $x \succsim y$).

so that the coalition S can improve upon the allocation, a contradiction to the fact that this allocation is in $N(E_r)$. Now, we show that $x_2^r \gtrsim_{2r} x_1^r/(r-1)$ and $x_i^r \sim_r x_k^r$ for all $i, k \in \{3,\ldots, r+2\}$. Suppose indeed that we have either (i) $x_1^r/(r-1) \succ_{2r} x_2^r$, (ii) $\exists$ i and k for whom $x_k^r \succ_r x_i^r$, or (iii) both (i) and (ii) hold at the same time. Define i^* by the condition $x_i^r \gtrsim_r x_{i^*}^r$ for all $i \in \{3,\ldots, r+2\}$. Consider the coalition S defined by $S = \{2 \cup i^*\}$ and define

$$y_2^r = \frac{1}{r} x_2^r + \left(1 - \frac{1}{r}\right)\frac{x_1^r}{r-1},$$

$$y_{i^*}^r = \frac{1}{r}\sum_{i=3}^{r+2} x_i^r.$$

We observe that

$$y_2^r + y_{i^*}^r = \frac{1}{r}(x_2^r + x_1^r) + \frac{1}{r}\sum_{i=3}^{r+2} x_i^r = \frac{1}{r}\sum_{i=1}^{r+2} x_i^r$$

$$= \frac{1}{r}(w + w') = w_2^r + w_{i^*}^r.$$

Furthermore, by strict convexity of $\gtrsim_{2r}$ and $\gtrsim_r$, should one of the alternatives (i), (ii), or (iii) hold, the coalition S could improve upon the allocation, a contradiction. We have thus proved that $x_1^r/(r-1) \sim_{2r} x_2^r$. By (3), we also have $x_1^r \sim_{1r} x_2^r(r-1)$. Then condition (3), strict convexity, and Pareto optimality of $(x_1^r,\ldots, x_{r+2}^r)$ imply that $x_1^r/(r-1) \sim_{2r} x_2^r$ and $x_1^r \sim_{1r} x_2^r(r-1)$ can hold simultaneously only if $x_1^r = x_2^r(r-1)$. Consequently, defining $x = x_1^r + x_2^r$, we have that $x_1^r = x(1-(1/r))$ and $x_2^r = x/r$. Furthermore, $x_i^r \sim_r x_k^r$ for all $i, k \in \{3,\ldots, r+2\}$ implies, with strict convexity of $\gtrsim_r$, that $x_i^r = x'/r$ for all $i, k \in \{3,\ldots, r+2\}$, if x' is defined by $x' = \sum_{i=3}^{r+2} x_i^r$. Q.E.D.

By the lemma, any allocation in the core of E_r can be represented by two vectors x and x'. Clearly all competitive allocations of E_r share this property. We establish now that only the competitive allocations remain in the core of E_r for all r.

Proposition 2. *If an allocation (x, x') is in $N(E_r)$ for all r, then it is competitive for all r.*

Proof. Let (x, x') be in $N(E_r)$ for all r. First, it is clear that x and x' are both in the interior of R_+^l. Suppose indeed, for instance, that x' is on the boundary of R_+^l. Then by assumption, x' is indifferent to 0. However, we must have $x' \gtrsim w' \succ 0$, where $x' \gtrsim w'$ follows from the fact that (x, x') is in $N(E_r)$ for all r, and $w' \succ 0$ follows from $w' \gg 0$ and monotonicity. Since $\gtrsim_1$

and $\succsim$ are furthermore strictly convex and representable by C^1 utility functions, there exists a unique price vector p which "sustains" (x, x') in E_1, i.e.,

$$y \underset{1}{\succ} x \Rightarrow p \cdot y > p \cdot x \tag{5}$$

and

$$y \succ x' \Rightarrow p \cdot y > p \cdot x'. \tag{6}$$

First, the price vector p also sustains (x, x') in E_r for all r. Indeed suppose for instance that, in E_r, $z \succ_r x'/r$; then by (4), $rz \succ x'$, which implies, by (6), that $p \cdot rz > p \cdot x'$, or $p \cdot z > p \cdot x'/r$. Accordingly, if we prove that, for all r, we must have $p \cdot (x'/r) = p \cdot (w'/r)$ and $p \cdot (x/r) = p \cdot (w/r)$ (which implies $p \cdot x(1 - (1/r)) = p \cdot w(1 - (1/r))$), we shall have completed the proof that the pair $[p, (x, x')]$ is a competitive equilibrium. So assume first that $p \cdot (x'/r) < p \cdot (w'/r)$. Then $p \cdot x' < p \cdot w'$. Since x' is interior to $R_+{}^l$ and $x' \succsim w'$, the differentiability assumption implies that there exists an integer ρ such that $(1 - (1/\rho))\, x' + (1/\rho)\, w' \succ x'$. Then it follows from (4) that in the economy E_ρ,

$$\frac{1}{\rho}\left[\left(1 - \frac{1}{\rho}\right) x' + \frac{1}{\rho} w'\right] \underset{\rho}{\succ} \frac{x'}{\rho}.$$

Consider then the coalition V in E_ρ consisting of $\{3, \ldots, \rho + 2\}$ and trader 1. Define

$$y_1 = \left(1 - \frac{1}{\rho}\right) x;$$

$$y_i = \left(1 - \frac{1}{\rho}\right) \frac{x'}{\rho} + \frac{1}{\rho} \frac{w'}{\rho}; \qquad i = 3, \ldots, \rho + 2.$$

We verify that

$$\sum_{i \in V} y_i = \left(1 - \frac{1}{\rho}\right) x + \left(1 - \frac{1}{\rho}\right) x' + \frac{w'}{\rho}$$
$$= \left(1 - \frac{1}{\rho}\right)(w + w') + \frac{w'}{\rho} = \sum_{i \in V} w_i{}^\rho,$$

so that V can improve upon (x, x'), a contradiction to the fact that $(x, x') \in N(E_\rho)$. Consequently we have proved that $p \cdot (x'/r) \geqslant p \cdot (w'/r)$. Assume now that $p \cdot (x/r) < p \cdot (w/r)$. Using an argument similar to that above, $x \succsim_1 w$ and $p \cdot x < p \cdot w \Rightarrow \exists \rho$ such that $(1 - (1/\rho))\, x + (1/\rho)\, w \succ_1 x$. Then it follows from (3) that, in the economy E_ρ,

$$y_1 \underset{\text{def}}{=} \frac{\rho - 1}{\rho}\left[\left(1 - \frac{1}{\rho}\right) x + \frac{1}{\rho} w\right] \underset{1\rho}{\succ} \frac{\rho - 1}{\rho} x = x_1{}^\rho,$$

$$y_2 \underset{\text{def}}{=} \frac{1}{\rho}\left[\left(1 - \frac{1}{\rho}\right) x + \frac{1}{\rho} w\right] \underset{2\rho}{\succ} \frac{x}{\rho} = x_2{}^\rho.$$

Consider the coalition V in E_ρ consisting of $\{3,\ldots,\rho+1\}$ and traders 1 and 2. We easily verify that

$$\sum_{i\in V} y_i = \left(1-\frac{1}{\rho}\right)x + \frac{w}{\rho} + \left(1-\frac{1}{\rho}\right)x' = \sum_{i\in V} w_i^\rho,$$

so that V can improve (x, x'), a contradiction. Consequently, for all r,

$$p\cdot\frac{x}{r} \geqslant p\cdot\frac{w}{r}.$$

Since (x, x') is an allocation, $p\cdot(x' \mid r) \geqslant p\cdot(w' \mid r)$ and $p\cdot(x \mid r) > p\cdot(w \mid r)$ together imply that, for all r,

$$p\cdot\frac{x'}{r} = p\cdot\frac{w'}{r} \qquad \text{and} \qquad p\cdot\frac{x}{r} = p\cdot\frac{w}{r},$$

which completes the proof that $[p, (x, x')]$ is a competitive equilibrium. Q.E.D.

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Candidates with Policy Preferences: A Dynamic Model

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The extensive literature concerned with formal models of political candidate strategies has almost without exception viewed policy as a means to winning.[1] In this paper, I suggest that the reverse is true--that candidates view winning as a means to policy. The paper argues that this perspective is more realistic in its assumptions and more accurate in its predictions. The approach is shown to yield considerable insight into a variety of situations with differing assumptions concerning voters and elections, including a sequential path analysis with no restrictions on the distribution of voter preferences; and a simultaneous presentation of platform positions with certain restrictions on the distribution of preferences.

The path analysis was originally put forward by Downs [3]. The winner in election T (the incumbent) must present the same platform in election $T+1$. The opposition is unrestricted in his choice of platform. With intransitivity, the incumbent will always lose and there will be a rotation in office. Both Buchanan [2] and Tullock [9] have argued that when the utility functions are type 1 (the indifference curves being concentric circles centered on the most preferred position of the voter, utility being solely a function of the Euclidian distance) the set of platforms chosen will eventually lead to the set of Pareto-optimal alternatives (the convex hull of the most preferred positions). However, McKelvey [6] has shown that majority rule cycles can go almost anywhere in the set of feasible choices and that the path does not converge to the Pareto-optimal set. In fact, it is easy to construct a path that expands ever outward from the Pareto-optimal set by making the winning position slightly closer to a majority of voters' preferred positions and as far away as possible from the preferred positions of a minority of voters, the majority not being identical in each step. If this characterization of the political process is correct, the implications are profound, as there is then no

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[1] The main exceptions are Wittman [10–12] and Barry [1], who characterize the electoral process in a different way (for example, only one issue is assumed and an electoral path is not considered). They also use less-formal arguments. The view of winning as the ultimate goal goes back to Smithies [8] and Schumpeter [7], and includes work by Downs [3], Kramer [5], and Hinich, Ledyard, and Ordeshook [4].

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relationship whatsoever between the preferences of the participants and the set of possible outcomes.

Kramer [6] suggested that candidates are not just interested in winning but in maximizing the number of votes. By considering a different hypothesis from Buchanan and Tullock concerning candidate motivation, he is able to show that the path converges to a subset of the Pareto-optimal set. In a remarkable proof, he demonstrated that the path will approach a set N^*. Looking at the intersection of all Pareto-optimal sets composed of V voters ($V = 1, 2, \ldots, N$), N^* is that nonempty intersection of Pareto-optimal sets with the smallest V. The incumbent will still always lose, but the opposition will tend to move closer to N^*. The underlying structure of the argument can readily be shown when $N = N^*$. Let the incumbent's position be outside the Pareto-optimal set composed of all N voters. There exists a point in the Pareto-optimal set preferred to the incumbent's position by all N voters. The opposition need not choose a position within this Pareto-optimal set, but if he does not, the position he does choose must also be preferred by N voters to the incumbent's position as the candidate maximizes votes. A position is preferred by N voters only if the position is closer to the N voters' Pareto-optimal set. Thus, the opposition will approach the Pareto-optimal set composed of N voters.

It should be noted that closer need not mean convergence to the set. For example, if each candidate in turn moves closer by amount $\epsilon/2^t$ (where t stands for the election, $t = 1, 2, \ldots$) to the Pareto-optimal set composed of N voters, as $t \to \infty$ the total movement towards this set will only approach ϵ. If this "degenerate" case is disallowed, then the candidates will actually enter or come close to the set N^*.

In this paper it is assumed that given a choice of winning platforms, the candidate will choose that platform which coincides most with his preferences. Certainly, this is a more reasonable assumption than vote maximization, for there is little if any payoff to getting more votes than are necessary for winning the election. In fact, vote maximization has the perverse result that the only participants not interested in policy are the candidates.[2] With imperfect information, a candidate who was interested in policy would again choose the position closest to his own preferred position from a set of positions with equal probability of winning, and would be willing to trade off the probability of winning for a desired platform position if this increased his expected utility (where utility is solely a function of which policy is implemented). Thus, concern for policy is a reasonable assumption under a variety of circumstances.

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[2] The rationale for vote maximization is due to an incorrect analogy from the economic sphere. Businessmen are not interested in what kind of chair they produce; therefore by analogy candidates should not be concerned with policy. However, this analogy is incorrect for two reasons. In the first place, the consumption of a chair is a private good while government policy is a public good, and therefore the candidate would be concerned with the latter. In the second place, the businessman in not interested in maximizing sales of chairs but rather in maximizing profits. Selling chairs is a means to an end in the same way that gaining office (getting votes) is a means to the end of preferred policy implementation.

Furthermore, it has considerable empirical support. President Johnson was willing to undertake an unpopular position concerning the war in Vietnam. In fact, he gave up the chance for reelection rather than give up on the policy which he felt was correct. Both Barry Goldwater and George McGovern were clearly concerned with policy. But even a candidate who is more "politically expedient" still has policy preferences. The candidate is willing to undertake policies which he finds slightly distasteful, if this greatly enhances his chances for election or reelection. On the other hand, the candidate will be less willing to compromise on an issue which is important to him when the compromise will only slightly increase his chance of winning. An investigation of the alternative model will now be undertaken.

(1) Let M_i be a vector of preferred positions of voter i in the Euclidean K space R^k. $i = 1, 2, \dots, N + 2$. N is odd. Let $M_x = M_{N+1}$, $M_y = M_{N+2}$ be a vector of preferred positions of candidates X and Y, respectively. The set of feasible points is a compact convex body in R.

(2) Let vR_iw if i prefers v to w or i is indifferent between v and w. R_i is complete. vP_iw if vR_iw and not wR_iv. v, w are K-dimensional vectors.

(3) i has a preference ordering over the points in R^k represented by a type 1 utility function; i.e., $vR_iw \Leftrightarrow \| v - M_i \| \leqslant \| w - M_i \|$, where $\| z \| = (\sum_{i=1}^{K} z_i^2)^{1/2}$.

(4) i votes for X if xP_iy; votes for Y if yP_ix; and votes for the incumbent if yR_ix and xR_iy. For expositional clarity assume that each candidate always votes for himself.

(5) If $X(Y)$ is the winning candidate in election t, then $X_{t+1} = X_t$ $(Y_{t+1} = Y_t)$; i.e., the incumbent (winner) from election t presents the same platform in election $t + 1$.

(6) w is a winning position for the opposition if a majority of voters strictly prefer w to the incumbent's position.

(7) Each candidate is assumed to maximize his expected utility in election t. Because a voter votes for X either with probability 1 or 0, this condition is identical to the following: From the set of winning positions the opposition chooses that w' preferred by himself to all other winning positions. If this position is not preferred to the incumbent's position, the opposition chooses his own most preferred position. If there is no unique

best position, he will choose from this subset that position which will lead to the highest preference in the next election; and iterate this process until there is a unique best solution. If there is no unique best position, he will choose from this subset of indifferent positions the "highest ranked" indifferent winning position, the "ranking" being transitive and complete, with all relationships being strict.

(8) Let S_j be a set of $(N + 1)/2$ most preferred positions. There are $\binom{N}{(N+1)/2}$ distinct S_j.

(9) S_j is a minimax set for X, S_x if for every set S_i there exists an M_i such that $\| M_i - M_x \| \geq \| M_k - M_x \|$ for all $k \in S_x$. S_y, a minimax set for Y, is defined in the same way.

(10) $B_x(B_y)$ is the minimax ball if it is the smallest ball with center $M_x(M_y)$ containing the most preferred points of $S_x(S_y)$ (see Fig. 1).

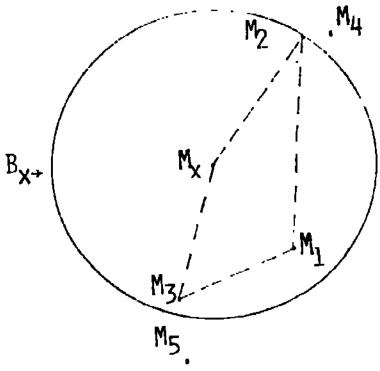

FIG. 1. B_x is the minimax ball. The dotted line is the Pareto-optimal set defined by S_x and M_x.

PROPOSITION 1. *Every $X_t(Y_t)$ $t \geq 1$ is in the minimax ball $B_x(B_y)$.*[3]

Proof. Assume Y is the incumbent. If $Y_{t-1} = Y_t$ is not an element of the Pareto-optimal set defined by S_x and M_x, then there exists an X' in the Pareto-optimal set strictly preferred by all of the members of the set, including X. Since this set has at least a majority of voters, X' is a winning position over Y_t. If X does not choose a point in this set, by assumption he must choose a winning position X^* that is not farther away from M_x than X' and therefore $X^* \subseteq B_x$. If $Y_{t-1} = Y_t$ is an element of the Pareto-optimal set defined by S_x and M_x and if there exists a point X' strictly preferred by X and a majority of voters to Y_t, then $X' \subseteq B_x$. If a point X' does not exist X will choose M_x.

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[3] Proposition 1 involves assumptions identical to those used by Kramer [5] except for the goal of the candidates. With slight alterations, similar results are obtained if voters vote for the incumbent, with probability P if indifferent. If $P = 0$, then the results are more restrictive.

If the candidates are only interested in winning there is no relationship between preferences of the voters and the set of electoral outcomes. However, when the candidates maximize votes, the preferences of voters do determine the electoral path. If the candidates have policy preferences, the preferences of the voters still influence the outcome, although the preferences of the candidates play a major role. Depending on the particular configuration of voter and candidate preferences the sets B_x and B_y may be smaller or larger than N^*. (However, if "degenerate" paths are considered, B_x and B_y are always more restrictive than the set of limits of vote maximizing paths.)

To best appreciate the difference between vote maximization and maximizing policy outcome, it is useful to look at the *bete noire* of transitivity—income distribution. In this case the indifference curves are no longer type 1. If all participants are purely selfish and each coordinate stands for income for a certain voter (or candidate), then the indifference curves are hyperplane perpendicular to the coordinate for the individual's income. For vote maximizing candidates, N^* is almost every conceivable income distribution where all income is distributed. For the income maximizing (policy preferring) candidates, the results are clear cut. In order to demonstrate this, the following terminology will be introduced.

(11) An equilibrium pair X^*, Y^* is said to exist if $X_t = X^*$, $Y_t = Y^*$ for all $t \geq 0$.

(12) An electoral path approaches an equilibrium pair X^*, Y^* if for all $\delta > 0$ there exists a t^* such that for all $t \geq t^*$, $\| X_t - X^* \| + \| Y_t - Y^* \| < \delta$.

(13) Let the amount of income that X offers to i in period $t = X_t^i$, $X_t^i \geq 0$.

Proposition 2. *Let the election be solely concerned with income distribution with the amount of income to be distributed being a fixed amount I; if $Y_t^x \neq I$, $X_t^y \neq I$ then the electoral path approaches a unique equilibrium pair X^*, Y^*, where $X^{x*} = I$, $X^{i*} = 0$ for $i \neq x$; $Y^{y*} = I$, $Y^{i*} = 0$ for $i \neq y$.*

Proof. Let Y be the incumbent in election t. Then $X_t^i = Y_t^i + \epsilon$ ($\epsilon > 0$) for the $(N + 1)/2$ smallest Y_t^i not including $Y_t^x < I$. X_t^x will receive the remainder. $X_t^i = 0$ for the remaining $(N - 1)/2$ individuals. In election $t + 1$, Y will offer ϵ to these $(N - 1)/2$ individuals; i.e., $Y_{t+1}^i = \epsilon$ for the $(N - 1)/2$ individuals for whom $X_t^i = 0$. Of the remaining X_t^i ($i \neq y$), Y will offer $Y_{t+1}^i = X_t^i + \epsilon$ to the smallest X_t^i. Y_{t+1}^y will receive the remainder. The other $Y_{t+1}^i = 0$. In election $t + 2$, X will offer ϵ to those $(N - 1)/2$ voters who received $Y^i_{t+1} = 0$, 2ϵ to one of the voters for whom $Y^i_{t+1} = \epsilon$, and the remainder to himself. This process will continue and for all $t > 0$,

$$\|X_{t+3} - X^*\| + \|Y_{t+3} - Y^*\| \leq \|X^x_{t+3} - I\| + \sum_{\substack{i=1\\ i\neq x}}^{N+2} \|X^i_{t+3} - 0\|$$

$$+ \|Y^y_{t+3} - I\| + \sum_{\substack{i=1\\ i\neq y}}^{N+2} \|Y^i_{t+3} - 0\| < \delta$$

for arbitrarily small ϵ ($\delta > NE + 3E$).

Thus almost all of the income goes to the two candidates.[4]

Proposition 2 demonstrates a great difference in form and substance between vote maximization and maximizing the utility from policy outcome. When the issue is income distribution, vote maximization leads to a set only slightly smaller than the feasible set. In contrast, preference for policy leads to a unique equilibrium pair. With regard to substance, vote maximization is typically seen as a method of registering "majority will" the benevolent invisible hand applied to the political process. If the candidates are vote maximizers, there exists no other position which is preferred by a greater number of voters to the set N^*. However, competitive policy maximizing candidates need not present positions which in any way reflect the "majority will" of the voters. In the income distribution case all but one of the candidates would strongly prefer that the income be divided equally. While the real world is not quite so dramatic, the analysis gives insight into why there is little redistribution of income and why many claim they do not have a real choice in the electoral process.

(14) Let the utility function of X be of the following nature: U_x is solely a function of the winning platform Z. $U_x(\|z - M_x\|)$ is twice differentiable, with

$$\frac{dU_x}{d\|Z - M_x\|} < 0, \qquad \frac{d^2U_x}{d\|Z - M_x\|^2} < 0.$$

Let the utility function of Y be a similar form.

Proposition 3. *If for all X_T, Y_T ($X_T \neq Y_T$) the probability that X_T will win the election is less than* 1 *and greater than* 0, *then there always exists a set of platforms X_T', Y_T' such that $X_T' = Y_T'$ and the expected utilities of both X and Y are higher.*

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[4] The lower the barriers to entry of a new candidate, the more the results will be mitigated as the candidates will offer income to the potential candidates. It should be noted that all results in the literature on candidate strategies are invalidated when there are low barriers to entry.

The proof is obvious due to the assumption of concavity. It should be noted that this proposition makes no assumptions concerning the indifference maps of the voters *nor does it make restrictions on the incumbent's choice of position.*

Similarity of the candidates' positions results when the candidates maximize votes and the feasible set is transitive. However, by Proposition 3, only identical positions are (*ex ante*) Pareto optimal for candidates with policy preferences. Where competition would lead the candidates to dissimilar platforms, it pays candidates with policy preferences to collude and present similar platforms. Therefore similarity of the candidates' positions cannot be used as proof of competition, vote maximization, or transitivity. Proposition 3 has implications somewhat similar to those of Proposition 2 (and thereby shows the robustness of the results). The presence of two candidates does not guarantee a competitive outcome. Consequently, the candidates need not respond to the interests of the voters. For example, in an election that is only concerned with income distribution, the candidates might collude to split all the income among themselves. Bipartisanship in the United States and restricted electoral competition among the oligarchy in Central America may be examples of this phenomenon.

Of course, there may be other reasons why the candidates present identical platforms, as Propositions 4 and 5 will show. These two propositions consider interesting "limit" cases. In Proposition 4, the candidates' preferences are diametrically "opposed" to those of the voters, while in the first part of Proposition 5 the candidates' positions are in diametric opposition to each other. It is no longer assumed that $X_t = X_{t-1}$ if X is the incumbent in election t, nor in Proposition 4 that the voters' preference functions are type 1.

PROPOSITION 4. *If $M_x = M_y$ and the candidates present their positions simultaneously, then $X_t = Y_t = M_x$ is stable equilibrium point.*

The proof is obvious.

PROPOSITION 5. *If the indifference curves are type 1, there exists a point M_m which is the most preferred position of voter m, and all the remaining voters can be put in pairs such that for each pair the indifference curves through M_m are separated by a hyperplane through M_m, then the following is true.*

(A) *If the indifference curves through M_m of X and Y are separated by a hyperplane through M_m, and M_x, $M_y \neq M_m$, then M_m is a minimax equilibrium if randomized strategies are not allowed.*[5]

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[5] In the context of a political campaign, it is not reasonable to assume randomized strategies. For example, one of the candidates (most likely, the incumbent) may have to announce his (nonrandom) position first. A platform which involves lotteries is also ruled out if the voters' utility functions are concave.

(B) *If the indifference curves through M_m of X and Y are on the same side of a hyperplane through M_m, then $M_x(M_y)$ is a minimax position of X(Y) if $M_x(M_y)$ is closer to M_m than $M_y(M_x)$.*

It should be noted that for type 1 indifference curves, the conditions in the proposition are both necessary and sufficient for a majority winner. Minimax means the worst the first candidate can do given that the second candidate maximizes his utility given the first candidate's position.

Proof of A. (See Fig. 2.)

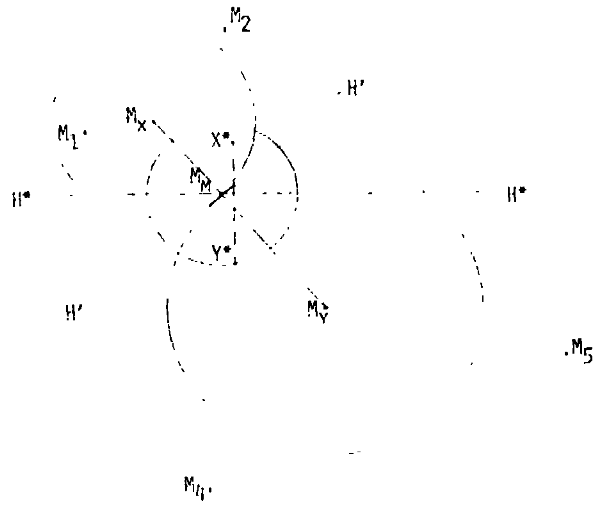

FIG. 2. If $Y^* \neq M_m$, then there exists an X^* preferred by X and a majority of voters to Y^*. X's and Y's indifference curves through M_m are separated by the hyperplane H'. The indifference curves of voters 1 and 5 (2 and 3) through M_m separated by another hyperplane (not drawn). X^* is within voter m's indifference curve through Y^*. H^* is the hyperplane through M_m perpendicular to the line X^*Y^*.

Let Y choose a point Y^* ($Y^* \neq M_m$); then there always exists an arbitrarily small open ball centered on M_m such that every point within the ball is strictly preferred by a majority of voters.

Let $\|Y^* - M_m\| = C$; let X^* be a point such that $\|X^* - M_m\| < C$. Define a hyperplane H^* through M_m which is perpendicular to the line connecting X^* and Y^*. If X^* and Y^* are in the same half-space, then a majority of voters in the other half-space strictly prefer X^* to Y^* as X^* is closer to every voter in this half-space, and every half-space through M_m contains a majority of voters by assumption. If X^* and Y^* are not in the same half-space, then all of the voters in the half-space which includes X^* strictly prefer X^* to Y^*. If Y^* is in the indifference curve of Y through M_m

(or more generally in the half-space H' which includes M_y and is defined by the hyperplane through M_m and perpendicular to the line connecting M_x, M_y, and M_m), then there exists a point X^* strictly preferred by X to M_m and Y^* and strictly preferred by a majority of voters to Y^*. Y strictly prefers M_m to this point. If Y^* is not in H', it is obvious that Y strictly prefers M_m to Y^* and also that Y prefers M_m to X's best choice against Y^*. Since M_m is strictly preferred by a majority of voters to any other point, M_m is a minimax position.

Proof of (B). (See Fig. 3.)

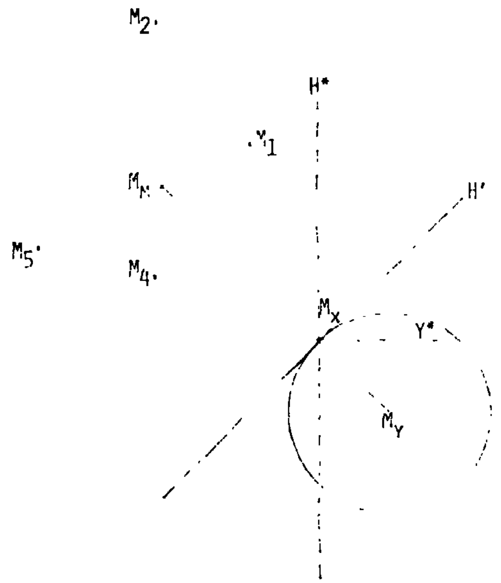

FIG. 3. If M_x is closer to M_m than M_y is to M_m, then M_x is a minimax position for X. Y prefers Y^* to M_x, but a majority of voters prefer M_x to Y^*.

Assume $\| M_x - M_m \| < \| M_y - M_m \|$. By assumption, M_x, M_y, and M_m are on a line. Let Y^* be a point closer to M_y than M_x is to M_y. Then Y^* is in the half-space H' not including M_m defined by the hyperplane through M_x and perpendicular to the line connecting M_x and M_y. The hyperplane through M_x and perpendicular to the line connecting M_x and Y^* creates another half-space H^* which includes M_m. All the voters in this half-space prefer M_x to Y^*, and there are at least a majority of voters in this half-space.

We may alter the rules so that Y will also choose M_x in this case; i.e., if there is no winning position preferred to the other candidate's minimax position, then the candidate will choose the other candidate's minimax position so that he will still be considered a viable candidate. (Both candidates will have a chance of winning if indifferent voters vote for a candidate with probability greater than 0 but less than 1.) Both candidates will then look alike, although they are not at the median voter's preferred position. This may explain those cases where both candidates are to "the left" or "right" of the majority.

CONCLUSION

This paper has shown the usefulness of assuming that candidates have policy goals. It suggests that in order to predict which government policies will be implemented, it is not only necessary to know the voters' preferences, but the candidates' preferences as well.

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The Eventual Failure of Price Fixing Schemes

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This essay examines the feasibility of a government buffer stock program which attempts to fix a relative price over time. In the model, *m* risk averse agents with random endowments of two commodities maximize expected utility over an infinite horizon. It is shown that a price fixing scheme will fail with probability 1, regardless of the price set by the government and regardless of the initial level of buffer stocks. The proof of this proposition turns on some well-known properties of random walks.

1. INTRODUCTION

In actual economies, uncertainty as to the direction and extent of future price movements is pervasive. This has led some policymakers to argue that government programs which attempt to fix prices could reduce uncertainty and increase economic welfare. This argument has been particularly prevalent in policy discussions concerning the stabilization of raw material and agricultural commodity prices and the stabilization of exchange rates of internationally traded currencies. With regard to commodity price stabilization, Keynes [6] argued for the establishment of an ever-normal granary to eliminate the violence of raw material price fluctuations associated with an unregulated competitive system. Other economists respond with the long-held belief that prices play a crucial role in allocating resources efficiently. Waugh [10] and Oi [7] have even argued that price instability may be beneficial, though Samuelson [8] showed that the fluctuations proposed in a partial equilibrium setting were not really feasible in a closed model. With regard to

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currency price stabilization, the debate over fixed versus flexible exchange rates has had a long history. The argument has centered in particular on whether speculation is or is not destabilizing and on whether there is or is not an adequate hedge against price uncertainty (see, for example, [1, 9]).

To some extent confusion on these issues stems from the failure of economists to provide coherent stochastic models. Many economic models are deterministic and hence give the impression that uncertainty is an anomaly rather than a fact of life. As for stochastic models, many are partial equilibrium and hence leave open the question of whether the uncertainty can be eliminated. This essay takes a modest step toward the formulation of a coherent general equilibrium model in which one can analyze price fixing schemes.

The model adopted is a pure exchange economy in which the endowments of two goods, (x) and (y), are stochastic. In each period the aggregate endowments of (x) and (y) are independently distributed, and each series is assumed to be independent and identically distributed over time. Both goods can be stored with no storage costs and no depreciation. Acting competitively, each of m risk averse individuals maximizes expected utility over an infinite horizon by choice of the amount of each good to consume in each period and the amount to carry over to the following period. The rate at which (y) exchange for (x) in each period is the price upon which the analysis focuses. It is supposed that the government attempts to maintain a fixed price by its willingness to exchange (y) for (x) with the public at a specified rate. In general such a policy requires that the government maintain buffer stocks of both goods.

For the specific model examined it turns out that regardless of the price set by the government and regardless of the initial level of government buffer stocks, a price fixing scheme fails in finite time with probability one. Equivalently, in the same model without government, maximizing individuals in competitive markets will not act in such a way as to fix the relative price over time. The relative price does play a crucial role in the allocation of resources over time. For example, with random aggregate endowments, there will be eventually a succession of realizations with the aggregate endowment of (y) low relative to the aggregate endowment of (x). In competitive markets without government the relative price of (y) would rise, thereby discouraging its consumption. If the relative price of (y) were fixed at its average value or less, the government's stocks of (y) would be depleted. In short, there must be some price flexibility; with the relative price fixed the government must absorb all disturbances in relative supply with its buffer stocks, yet it is not capable of doing so.

This essay proceeds as follows. Section 2 describes the structure of the model and analyzes the decision problem of the individual with a fixed relative price. The properties of individual storage and consumption decisions are then shown to be inconsistent with any initial level of government stocks and with any fixed price. Formal proofs are contained in Section 3. Section 4 presents some concluding remarks.

2. STRUCTURE AND PROPERTIES OF THE MODEL

The stochastic nature of the endowments is now made precise. In each time period there is a finite set of states Ω. Let P be a probability measure with $0 < P(\omega)$ for each $\omega \in \Omega$. Let X_t and Y_t denote the economy's total endowment of (x) and (y), respectively, at time t. X_t and Y_t are assumed to be independent, nonnegative random variables. Moreover, X_t and Y_t are each independent and identically distributed over time. It is also assumed that there exists some $\bar{\omega} \in \Omega$ such that $X_t(\bar{\omega}) = Y_t(\bar{\omega}) = 0$. Let $X_t = E(X_t) + \epsilon_{xt}$ and $Y_t = E(Y_t) + \epsilon_{yt}$, where ϵ_{xt} and ϵ_{yt} are each symmetric and take on values in the integer lattice.

Each individual is assumed to have a fixed and constant proportion of the aggregate endowment of each good, δ_x^j and δ_y^j being the jth individual's share of X_t and Y_t, respectively. Let $Z_t^{j\prime} = [\delta_x^j X_t, \delta_y^j Y_t]$. Let $K^j_{x,t}$ and $K^j_{y,t}$ denote the jth individual's stocks of (x) and (y), respectively, at the end of the period t, to be chosen during period t. Let X_t^j and Y_t^j denote the jth individual's consumption of (x) and (y), respectively, at time t. Let R_t denote the relative price of (y) in terms of (x) at time t. Let $r_t' = [1, R_t]$. Then, regarding as parameters current and future prices $\{R_t\,;\, t = 1, 2,\ldots\}$ and initial stocks $\{K^j_{x,0} \geqslant 0,\ K^j_{y,0} \geqslant 0\}$, the objective of individual j is to maximize $E_0 \sum_{t=1}^{\infty} \beta^{t-1} U^j(X_t^j, Y_t^j)$ with respect to $\{K^j_{x,t}, K^j_{y,t}, X_t^j, Y_t^j, t = 1, 2,\ldots\}$ subject to the constraints:

$$\text{(i)}\quad X_t^j + R_t Y_t^j \leqslant K^j_{x,t-1} + R_t K^j_{y,t-1} + r_t' Z_t^j - K^j_{x,t} - R_t K^j_{y,t}, \qquad t = 1, 2,\ldots,$$

$$\text{(ii)}\quad X_t^j \geqslant 0; \qquad Y_t^j \geqslant 0, \qquad t = 1, 2,\ldots,$$

$$\text{(iii)}\quad K^j_{x,t} \geqslant 0; \qquad K^j_{y,t} \geqslant 0, \qquad t = 1, 2,\ldots.$$

Here $0 < \beta < 1$, and $E_t(\cdot)$ denotes the expectation conditioned on all realizations up to and including time t. Constraint (i) is the budget constraint for individual j at time t; to be noted is the imposed absence of borrowing possibilities and forward contracts. Constraint (ii) implies that expenditures at time t must be nonnegative. Constraint (iii) states that the stocks of each good must be nonnegative.

$U^j(\cdot, \cdot)$ is assumed to have the following properties:

(i) $U^j(\cdot, \cdot) = g^j[H^j(\cdot, \cdot)]$, where $g^j(\cdot)$ is of class C^2 with first derivative $g_1^j(\cdot) > 0$ and second derivative $g_{11}^j(\cdot) < 0$. Also, $g^j(0) = 0$, $g_1^j(H) \to 0$ as $H \to \infty$, and $g_1^j(H) \to \infty$ as $H \to 0$.

(ii) $H^j(\cdot,\cdot)\colon R_+^2 \to R_+^1$ is of class C^2 with strictly positive first partial derivatives. Also, $H^j_{11}(\cdot,\cdot) < 0$ with the bordered Hessian of $H^j(\cdot,\cdot)$ positive.

(iii) $H^j(\cdot,\cdot)$ is homogeneous of degree 1.

(iv) $H_1^j(X_t^j, Y_t^j) \to \infty$ as $(X_t^j, Y_t^j) \to (0, \bar{Y})$ for each $\bar{Y} > 0$, $H_2^j(X_t^j, Y_t^j) \to \infty$ as $(X_t^j, Y_t^j) \to (\bar{X}, 0)$ for each $\bar{X} > 0$.

Thus $U^j(\cdot,\cdot)$ is a positive monotonic transformation of a homogenous function of degree 1 and is said to be homothetic. $U^j(\cdot,\cdot)$ possesses properties (ii) and (iv), among others.

Properties of individual storage decisions are now derived on the assumption that the government maintains a fixed price for all time. The assumption that price will remain fixed for all time does not eliminate uncertainty from the decision problem of the individual; endowments are still stochastic. Yet, each individual need be concerned only with the total value of stocks of (x) and (y) in terms of one of the goods, say (x). The exchange of (y) for a predetermined amount of (x) in any future period is guaranteed by the government. The individual's decision problem may be viewed in two stages. In a given period and state, the individual has available a known amount of savings in terms of (x) of the previous period and the realized value of his endowment in the specified state. He decides on his optimal amount of savings in terms of (x), and consequently on his current expenditures in terms of (x). With this timing problem solved, the individual then chooses his current consumption of (x) and (y) at the specified price.

To state this more formally, the following notation is needed. Let $\bar{R}$ denote the relative price of (y) in terms of (x) fixed by the government. Then, by assumption, $R_t = \bar{R}$ for all $t \geqslant 1$. Let $r' \in [1, \bar{R}]$. Let $K_t^j = K^j_{x,t} + \bar{R}K^j_{y,t}$. Let $I_t^j = X_t^j + \bar{R}Y_t^j$. Then, by the strict monotonicity of preferences, $I_t^j = K^j_{t-1} + r'Z_t^j - K_t^j$. By virtue of the separability of the objective function over time, maximizing choices of X_t^j and Y_t^j can be expressed as functions of I_t^j and $\bar{R}$ only. Let $h_x^j(I_t^j, \bar{R})$ and $h_y^j(I_t^j, \bar{R})$ denote maximizing choices of X_t^j and Y_t^j, respectively, for the function $U^j(X_t^j, Y_t^j)$ subject to the constraint that $X_t^j + \bar{R}Y_t^j = I_t^j$. Then an indirect utility function $V^j(\cdot,\cdot)$ is defined by $V^j(I_t^j, \bar{R}) = U^j[h_x^j(I_t^j, \bar{R}), h_y^j(I_t^j, \bar{R})]$. Upon substitution, given $K_0^j \geqslant 0$, the objective of individual j is to maximize $E_0 \sum_{t=1}^{\infty} \beta^{t-1} V^j(K^j_{t-1} + r'Z_t^j - K_t^j, \bar{R})$ with respect to $\{K_t^j;\ t = 1, 2,\ldots\}$ subject to the constraints:

(i) $K^j_{t-1} + r'Z_t^j - K_t^j \geqslant 0, \qquad t = 1, 2,\ldots,$

(ii) $K_t^j \geqslant 0, \qquad t = 1, 2,\ldots.$

Constraint (ii) is apparently weaker than the restriction that $K^j_{x,t}$ and $K^j_{y,t}$ each be nonnegative. However, the imposition of the latter constraint would not alter the optimal storage rules for $\{K_t^j;\ t = 1, 2,\ldots\}$.

From the homogeneity of $H^j(\cdot, \cdot)$, the demand functions $h_x^j(\cdot, \bar{R})$ and $h_y^j(\cdot, \bar{R})$ are linear. It follows that $V^j(\cdot, \bar{R})$ is strictly concave.[1] Clearly, $V^j(\cdot, \bar{R})$ is of class C^1 and strictly increasing. From property (i) of $U^j(\cdot, \cdot)$, $V_1^j(I, \bar{R}) \to \infty$ as $I \to 0$, and $V_1^j(I, \bar{R}) \to 0$ as $I \to \infty$.

Given a fixed relative price, the individual's stochastic dynamic programming problem is equivalent to a choice between consumption and savings in a one-good model in which future income is subject to random shocks. Foley and Hellwig [4] and Brock and Mirman [2] have established existence and analyzed optimal consumption policies for similar problems, and hence the crucial properties of the solution to the problem of this essay are given here without proof.[2] Let $W_t^j = K_{t-1}^j + r'Z_t^j$ denote the wealth of individual j at time t in terms of (x). Let $\tilde{K}_t^j = \varphi^j(W_t^j, \bar{R})$ denote a maximizing choice of stocks K_t as a function of W_t^j and $\bar{R}$. Also, let $M^j(\bar{R}) = \max_{\omega \in \Omega} r'Z_t^j(\omega)$. Then the following properties can be established:

(i) For every $\bar{R} > 0$ there exists a unique $K_*^j(\bar{R})$ such that

$$\varphi^j[K_*^j(\bar{R}) + M^j(\bar{R}), \bar{R}] = K_*^j(\bar{R}) > 0.$$

Also, $\varphi^j(0, \bar{R}) = 0$.

(ii) For every $\bar{R} > 0$ let $A^j(\bar{R}) = [0, K_*^j(\bar{R})]$. Then if $\tilde{K}_t^j \in A^j(\bar{R})$, $\tilde{K}_\tau^j \in A^j(\bar{R})$ for all $\tau > t$.

(iii) For all $K_0^j \geq 0$, prob $\{\tilde{K}_t^j > K_*^j(\bar{R}) \text{ i.o.}\} = 0$.

As $\tilde{K}_t^j$ eventually enters the set $A^j(\bar{R})$ with probability 1, it may be assumed $K_0^j \in A^j(\bar{R})$. Then $\tilde{K}_t^j$ is bounded above by $K_*^j(\bar{R})$ for all t.

Given the existence of a solution to the individual's optimization problem when the government is maintaining a fixed price, there will be determined some $\{I_t^j; j = 1, 2, \ldots m;\ t = 1, 2, \ldots\}$. Given I_t^j, individual j will purchase utility maximizing quantities of (x) and (y) at the fixed price $\bar{R}$. For the policy to be feasible it is specified that $\bar{R}$ be an equilibrium price in the sense that any excess demand for (x) by individuals be matched by an excess supply on the part of the government. It is now shown that no price can be maintained indefinitely far into the future, regardless of the initial level of government stocks.

The proof of this proposition turns on associating with each fixed price

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[1] $V^j(\cdot, \bar{R}) = g^j[\tilde{V}(\cdot, \bar{R})]$, where $\tilde{V}^j(\cdot, \bar{R})$, the indirect utility function for $H(\cdot, \cdot)$, is linear. Strict concavity of $V^j(\cdot, \bar{R})$ then follows from the strict concavity of $g^j(\cdot)$. Note also that this implies that $U^j(\cdot, \cdot)$ is strictly concave. Let D denote the bordered Hessian of $U^j(\cdot, \cdot)$. It can be shown that $V_{11}^j = (U_{11}^j U_{22}^j - U_{12}^j U_{21}^j)/(-D)$. By property (ii), $D > 0$ and so the numerator is positive. Also, $U_{11}^j = g_{11}^j (H_1^j)^2 + g_1^j H_{11}^j < 0$.

[2] The problem solved by Foley and Hellwig [4] has a two-point distribution of income in each period, but the generalization to the distribution of this essay is straightforward.

$\bar{R}$ set by the government a pure exchange, nonstochastic, no-storage economy for which $\bar{R}$ is a competitive equilibrium price. In some sense, the most difficult price for which to prove the infeasibility of price fixing is the competitive equilibrium price for the model with endowments $E(X_t)$ and $E(Y_t)$ for (x) and (y), respectively. Such a price corresponds to the competitive equilibrium price of the "average" economy in the model without storage but with random endowments. For this price, properties of a random walk in R^2 with zero drift are used. Roughly speaking, with probability 1 there will be runs over a number of periods in which there are smaller than average endowments of (y) and higher than average endowments of (x). Total incomes will be at their average values, and agents will wish to consume average amounts of (y) at the fixed price. But (y) is in short supply, and eventual failure is inevitable. For all other prices, properties of random walks with nonzero drift are used; if the government sets a price of (y) which is in some sense too low, there will be an excess supply of (x).

More formally, consider a static pure exchange economy with fixed endowments. Each agent j has the same utility function $U^j(\cdot, \cdot)$ of the stochastic model with storage. Also let $Z^j = [\delta_x^j X, \delta_y^j Y]$ denote the endowment of agent j, where the share parameters δ_x^j and δ_y^j are the same as in the stochastic model with storage. It is assumed that the $U^j(\cdot, \cdot)$ and Z^j, $j = 1, 2, \ldots m$, are such that (x) and (y) are gross substitutes. Then for $X > 0$ and $Y > 0$ there exists a unique competitive equilibrium price $R^* = \pi(X, Y)$, where $\pi(\cdot, \cdot) : (0, \infty)^2 \to R_+^1$. The following properties of $\pi(\cdot, \cdot)$ are established in Section 3.

LEMMA 1. *$\pi(\cdot, \cdot)$ is continuous.*

LEMMA 2. *$\pi(X, \bar{Y}) \to 0$ as $X \to 0$ for each $\bar{Y} > 0$. $\pi(\bar{X}, Y) \to \infty$ as $Y \to 0$ for each $\bar{X} > 0$.*

These then yield the following

PROPOSITION. *Given any fixed price $\bar{R} \in [0, \infty)$ and any initial level of government stocks of (x) and (y), there exists with probability 1 some $T^* < \infty$ at which those stocks will be insufficient to maintain the fixed price.*

The proposition has the obvious

COROLLARY. *If there exists a competitive equilibrium without government with relative price R_t^* at time t, then there does not exist some constant $\bar{R}$ such that $R_t^* = \bar{R}$ for all t.*

It might be argued that the expected time of failure of the price fixing scheme is infinite and that this mitigates the conclusions of the proposition. However, the proposition asserts that failure occurs with probability 1 in *finite* time.

3. FORMAL PROOFS

Proof of Lemma 1. Let $J(X, Y, R) = \sum_{j=1}^{m}[h_x{}^j(\delta_x{}^j X + \delta_y{}^j Y R, R) - \delta_x{}^j X]$. Then $R^* = \pi(X, Y)$ is chosen so that $J[X, Y, R^*] = 0$. By properties (i) and (iv) of $U^j(\cdot, \cdot)$ and the implicit function theorem, $h_x{}^j(\delta_x{}^j X + \delta_y{}^j Y R, R)$ is of class C^1 with respect to X, Y, and R. Hence, the first partial derivatives of $J(\cdot, \cdot, \cdot)$ are all continuous with respect to X, Y, and R. As (x) and (y) are gross substitutes, $J_3(\cdot, \cdot, \cdot) > 0$. Hence, the implicit function theorem applies and $\pi(\cdot, \cdot)$ is continuous. Q.E.D.

Proof of Lemma 2. The proof is by contradiction. Suppose that for some $\bar{Y} > 0$, $\pi(X, \bar{Y}) \not\to 0$ as $X \to 0$. Then there exists a sequence $\{X_n\}$ such that $X_n \to 0$ as $n \to \infty$ and for which the corresponding sequence of equilibrium prices, $\pi(X_n, \bar{Y})$, is bounded from below by some $R'' > 0$. Then as (x) and (y) are gross substitutes, $J[X_n, \bar{Y}, \pi(X_n, \bar{Y})] \geq J[X_n, \bar{Y}, R'']$. But

$$\lim_{n \to \infty} J[X_n, \bar{Y}, R''] = \sum_{j=1}^{m} h_x{}^j[\delta_y{}^j \bar{Y} R'', R''] > 0.$$

With $J[X_n, \bar{Y}, \pi(X_n, \bar{Y})] = 0$, this establishes the desired contradiction. By a symmetric argument, $\pi(\bar{X}, Y) \to \infty$ as $Y \to 0$ for each $\bar{X} > 0$. Q.E.D.

Proof of Proposition. There are four cases to be considered.

Case (i). $\bar{R} = 0$.

By property (ii) of $U^j(\cdot, \cdot)$, agents are never satiated with respect to consumption of (y). Hence if $\bar{R} = 0$, government stocks of (y) will be depleted in the first period of operation.

Case (ii). $\bar{R} = \pi[E(X_t), E(Y_t)]$.

Let G_0 denote the stock of (y) held by the government at $t = 0$, A_t denote the cumulative net amount of (y) sold to individuals by the government in exchange for (x) up through and including time t, and $Y_t{}^*$ denote the aggregate of (y) over all individuals made available for consumption from private wealth in period t. Also, let $S_{yT} = \sum_{t=1}^{T} \epsilon_{yt}$, $S_{xT} = \sum_{t=1}^{T} \epsilon_{xt}$, and $S_T = [S_{xT}, S_{yT}]$. Then S_T is a random walk in R^2 and has the property that for any integers B_x and B_y there exists some $T^* < \infty$ such that $S_{xT^*} = B_x$ and and $S_{yT^*} = B_y$ with probability one (see [5, p. 796]). For the proof let B_y be the largest integer less than or equal to $-G_0 - 1 - \sum_{j=1}^{m} (K_*{}^j/\bar{R})$, and let B_x be such that $\sum_{j=1}^{m} h_y{}^j(\delta_x{}^j B_x + \delta_y{}^j B_y \bar{R} - K_*{}^j) \geqslant 0$. This last choice is possible by the linearity of $h_y{}^j(\cdot, \bar{R})$. In period t the government must sell $\sum_{j=1}^{m} h_y{}^j(I_t{}^j, \bar{R}) - Y_t{}^*$ units of (y) to the public. Then the cumulative net sales of (y) at time T^* will be

$$A_{T^*} = \sum_{t=1}^{T^*} \left[\sum_{j=1}^{m} h_y{}^j(I_t{}^j, \bar{R}) - Y_t{}^*\right] = \sum_{j=1}^{m} h_y{}^j\left(\sum_{t=1}^{T^*} I_t{}^j, \bar{R}\right) - \sum_{t=1}^{T^*} Y_t{}^*,$$

using the linearity of $h_y^j(\cdot, \bar{R})$. Also,

$$\sum_{t=1}^{T^*} Y_t^* \leq \sum_{t=1}^{T^*} Y_t + \sum_{j=1}^{m} (K_*^j/\bar{R})$$

and

$$\sum_{t=1}^{T^*} I_t^j \geq \sum_{t=1}^{T^*} (\delta_x^j X_t + \delta_y^j Y_t \bar{R}) - K_*^j.$$

From these two inequalities and by substitution one obtains the inequality

$$\begin{aligned} A_{T^*} &\geq \sum_{j=1}^{m} \{h_y^j[\delta_x^j E(X_t)\, T^* + \delta_x^j S_{xT^*} + \delta_y^j E(Y_t)\, \bar{R} T^* + \delta_y^j S_{yT^*} \bar{R} - K_*^j, \bar{R}]\} \\ &\quad - E(Y_t)\, T^* - S_{yT^*} - \sum_{j=1}^{m} (K_*^j/\bar{R}) \\ &= T^* \left\{ \sum_{j=1}^{m} h_y^j[\delta_x^j E(X_t) + \delta_y^j E(Y_t)\, \bar{R}, \bar{R}] - E(Y_t) \right\} \\ &\quad + \sum_{j=1}^{m} h_y^j[\delta_x^j B_x + \delta_y^j B_y \bar{R} - K_*^j, \bar{R}) - B_y - \sum_{j=1}^{m} (K_*^j/\bar{R}). \end{aligned}$$

But by the choice of $\bar{R}$.

$$\sum_{j=1}^{m} h_y^j[\delta_x^j E(X_t) + \delta_y^j E(Y_t)\, \bar{R}, \bar{R}] - E(Y_t) = 0.$$

Hence by the choice of B_x and B_y, $A_{T^*} \geq G_0 + 1 > G_0$.

Case (iii). $0 < \bar{R} < \pi[E(X_t), E(Y_t)]$.

$\pi(\cdot, \cdot)$ is continuous and $\pi[X, E(Y_t)] \to 0$ as $X \to 0$. Hence, by the intermediate value theorem there exists some X^*, $0 < X^* < E(X_t)$, such that $\bar{R} = \pi[X^*, E(Y_t)]$. Let $\gamma = E(X_t) - X^*$. Then $X_t = X^* + (\gamma + \epsilon_{xt})$ with $E(\epsilon_{xt} + \gamma) > 0$. Let $S_{xT} = \sum_{t=1}^{T}(\epsilon_{xt} + \gamma)$ and let $S_{yT} = \sum_{t=1}^{T} \epsilon_{yt}$. As in Case (ii) let B_y be the largest integer less than or equal to $-G_0 - 1 - \sum_{j=1}^{m}(K_*^j/\bar{R})$. Then by Feller [3, pp. 202-203], S_{xt} visits $(-\infty, a)$ a finite number of times for all $a > 0$. Hence, there exists some T^* such that $S_{yT^*} \leq B_y$ and S_{xT^*} is large enough that $\sum_{j=1}^{m} h_y^j(\delta_x^j S_{xT^*} + \delta_y^j S_{yT^*} \bar{R} - K_*^j, \bar{R}) \geq 0$. Then as in Case (ii)

$$\begin{aligned} A_{T^*} &\geq T^* \left\{ \sum_{j=1}^{m} h_y^j[\delta_x^j X^* + \delta_y^j E(Y_t)\, \bar{R}, \bar{R}] - E(Y_t) \right\} \\ &\quad + \sum_{j=1}^{m} h_y^j(\delta_x^j S_{xT^*} + \delta_y^j S_{yT^*} \bar{R} - K_*^j, \bar{R}) - S_{yT^*} - \sum_{j=1}^{m} (K_*^j/\bar{R}) > G_0. \end{aligned}$$

Case (iv). $\pi[E(X_t), E(Y_t)] < \bar{R} < \infty$.

By a proof quite similar to Case (iii), initial stocks of (x) will be insufficient to maintain the fixed price. Q.E.D.

Proof of Corollary. Let $G_0 = 0$ and apply the proof of the proposition. Q.E.D.

4. CONCLUDING REMARKS

In the model the government attempts to fix a relative price over time by its willingness to exchange two commodities with the public at a specified rate. It has been shown that under specified assumptions such a policy fails with probability 1. Yet there remains the question of whether there exist feasible price fixing schemes based on endowment taxation. One particularly simple scheme based on commodity taxation also fails. For suppose that the government were to attempt to maintain the price corresponding to Case (ii) of the proposition by confiscation (restitution) of the surfeit (deficit) of X_t over $E(X_t)$ in each period, and similarly for (y). Note that if such a policy were successful there would be no storage by individuals. Then the cumulative net amount of (x) confiscated by the government through period T would be $\sum_{t=1}^{T} \epsilon_{xt}$, and the stated properties of random walks imply that government stocks eventually will be insufficient to maintain the fixed price. Other fixed prices fail similarly. However, it is difficult to establish the properties of more complicated price fixing schemes based on endowment taxation.

The failure of the most obvious price fixing schemes raises some questions concerning the existence and properties of a competitive equilibrium in the model without government. It can be shown that the set of economies which possess competitive equilibria without government is nonempty; hence the result of this essay is not vacuous. In particular, if all agents have identical preferences and endowments, then there exists a solution to the stochastic dynamic programming problem of the representative individual. (It can be noted in passing that this autarkic solution has the property that the implicit distribution of prices displays some smoothness as compared with the distribution which would prevail if storage by individuals were prohibited.) Yet the question of existence is a difficult one in a model of any generality.

Given existence, the properties of the competitive equilibrium path without government could be analyzed. If conventional results are any guide, it might be shown that the path is Pareto optimal. This then would raise some doubt concerning the desirability of even feasible government buffer stock programs. Yet these conjectures are beyond the scope of the present essay.

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Excess Supply Equilibria

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This paper addresses a fundamental problem in economic theory: How can there be equilibria of the economic system where some commodity is in excess supply, yet that commodity's relative price shows no tendency to fall? Of course, the principal example of such a phenomenon is an economy experiencing a prolonged period of involuntary unemployment of the labor force during which there is no significant change in the real wage.

In the following pages, I shall describe a two-commodity, general equilibrium model that has a continuum of unemployment equilibria, one for any given unemployment rate. The important feature of this model is that workers establish their wage rates in an attempt to maximize expected utility. The information upon which these wage setting decisions are based is provided by actual labor market transactions.

Despite the voluntary nature of the wage setting decision, I shall argue that each equilibrium of this economy exhibits involuntary unemployment in the Keynesian sense. For there will always be another equilibrium with a lower real wage, a higher level of employment, and at which (at least when workers are risk neutral) each worker achieves a higher level of expected utility.

The publication of Leijonhufvud's exegetical treatise, "On Keynesian Economics and the Economics of Keynes," marked a resurgence of economists' interest in an important and still unresolved theoretical controversy. The issue was first seriously debated after the appearance of Keynes' "General Theory." The question is this: Is it possible to have "equilibria" of the economic system where some commodity is in excess supply, yet that commodity's relative price, although positive, shows no tendency to fall?

Of course, the empirical phenomenon that stimulated Keynes' theoretical work and also the ensuing debate, was the tendency for capitalist economies to suffer prolonged periods of significant, involuntary unemployment of the labor force. From this perspective, the issue was, and is, whether or not such unemployment is an "equilibrium" phenomenon. In other words, can economic theory explain involuntary unemployment as a "steady state" that emerges from a consistent, choice theoretic specification of economic be-

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havior? Or must any such theory instead imply that, at least in the "long run," relative prices will adjust so as to clear all markets?

On one side of this issue are the theorists who rationalize involuntary unemployment as a consequence of economic disequilibrium. Such theories are generally of two types.

First, there are those models resting upon the assumption that the Walrasian auctioneer, or "invisible hand," is, at least in the short run, ineffective in moving prices to their Pareto-efficient, Walrasian equilibrium values. In these models, prices are treated as fixed and exogenous, while quantities transacted at these prices are allowed to vary so as to achieve an equilibrium of expectations. Such "fix-price" analyses do not attempt to explain why the vector of prices at which trading occurs will so often be "wrong," i.e., different from the Walrasian equilibrium prices. Seminal examples of this sort of model are contained in papers by Barro and Grossman [1] and Benassy [2].

An alternative "disequilibrium" interpretation of involuntary unemployment focuses explicitly upon the dynamic behavior of the economic system. These theories admit that the only "steady states" of the dynamic adjustment process may indeed be full-employment equilibria. But dynamic expectations–price–quantity interactions may cause the system to "spend most of its time" away from such equilibria. For a recent articulation of this viewpoint, see Tobin [4].

It seems fair to assert, however, that these particular interpretations of involuntary unemployment as a "disequilibrium" phenomenon are of relatively recent origin. The more orthodox view, exposited in virtually every textbook on the subject, is that such unemployment can persist even when the economy is in full equilibrium. From this perspective, the important puzzle is, then, the failure of the real wage to fall as a result of the excess supply of labor. This is usually explained by "rationalizing" money illusion on the part of workers; alternatively, the market power of labor unions or workers' desire to maintain wage differentials is identified as the wrench in the Walrasian price adjustment mechanism. In a somewhat different vein, some recent work by Salop [3] suggests that when wage rates are set by profit maximizing firms, some involuntary unemployment, arising from the presence of training costs for workers, may be consistent with full equilibrium.

In this paper I shall develop a model of macroeconomic activity that leads to a rather different explanation of involuntary unemployment. This theory starts from three simple observations about the world.

First of all, prices are set in most markets by individual economic agents so as to further their own economic interests. Second, these price setting decisions are usually guided by data and information generated by observed market transactions. Finally, market participants often have only limited information about their trading opportunities; consequently, the price setters in such markets will discover that they are not perfect competitors, i.e., the demand curves they confront have finite rather than infinite price elasticity.

In the following sections I shall incorporate these observations into a two-commodity, general equilibrium model that exhibits a continuum of unemployment equilibria. Unemployment rates in equilibrium can be arbitrarily small or as high as 100%, depending upon the level of real government expenditure.

It is important to ask why the real wage shows no tendency to fall at such unemployment equilibria. The answer reveals this theory's important features.

In this economy it is assumed that each worker determines the nominal wage rate at which he offers his labor. There are many firms hiring workers; each firm hires workers from its own "labor pool," assumed to be small relative to the total labor supply; the firm's labor pool is determined anew at the beginning of each trading period via a process of random search by workers for jobs. Low-wage workers in a firm's labor pool are hired before high-wage workers.

I shall show that in such a labor market all workers have a positive probability of being employed, even though they may be quoting different wage rates. Consequently, workers observe that they are not perfect competitors; instead, the probability of being employed is found to be a declining function of the wage rate. One defining property of equilibrium will be that the level of expected utility achieved by any worker is constant and independent of the wage he quotes. In particular, when the economy is in equilibrium, no worker sees any advantage to lowering his own wage quotation, despite a positive level of aggregate unemployment. Workers with lower wage quotations are seen to have higher employment rates but lower incomes when employed; these effects just offset one another in equilibrium.

This then is an example of a market where price setters make decisions based upon market generated data; because information is a costly commodity, price setters are not perfect competitors. Furthermore, the quantity sold at any price by an individual supplier is a random variable. In such circumstances, aggregate excess supply is seen to be consistent with "rational" pricing behavior. The Walrasian auctioneer has vanished from the marketplace.

There is a final issue to be raised. In what sense is the unemployment appearing in this model involuntary? After all, workers are setting their wage rates voluntarily. I shall argue that there is involuntary unemployment in the Keynesian sense: for any equilibrium, there is another equilibrium with a higher level of employment and a lower real wage. At least when workers are risk neutral, equilibria with high levels of employment are always Pareto superior to those with low levels of employment. In this sense, every equilibrium exhibits Keynesian involuntary unemployment.

1. CONSUMERS

This section describes the behavior of the worker–consumers in our hypothetical economy.

Throughout this paper I shall be assuming that there are only two commodities: leisure and output. Output is perishable and cannot be stored.

The first assumption I make is that there is a very large number, in fact a continuum, of consumers; for convenience we suppose that the "measure" of the set of all consumers is unity.

Next, suppose that each consumer has a flow endowment of L units of leisure per trading period. The utility a consumer derives from consuming c units of output and h units of leisure during a single trading period will be denoted by $u(c, h)$. We assume that u is a Von Neumann–Morgenstern utility function; furthermore, all consumers are assumed to have preferences that are convex and representable by the same function u.

I now assume that each consumer selects, at the start of each trading period, the nominal wage at which he will offer to sell his labor during that trading period. Different consumers may select different wage rates. Let $X(w)$ denote the "number" of consumers quoting nominal wage rates not exceeding w. Since the measure of the set of all consumers is assumed to be unity, we note that

$$0 \leq X(w) \leq 1.$$

I next assume that during any trading period, there is always an infinite number of wage rates at which some labor is being offered. This can be formalized by supposing that

$$X(w) = \int_0^w x(s)\, ds,$$

where $x(\cdot)$ is a Lebesgue measurable, real valued function that integrates to one. Thus $X(\cdot)$ is the wage distribution during a typical trading period, and $x(\cdot)$ is the corresponding wage density.

Two questions must now be addressed. First, given that a consumer quotes a wage of w during a trading period, how much labor does he offer at that wage? Second, what factors determine which wage rate a consumer quotes during a trading period?

To answer the first, it will be convenient to make a particular choice of units in which to measure labor. It will eventually be assumed that the production possibilities set for this economy is strictly convex; since all consumers are identical and have convex preferences, there is a unique Pareto optimum for this economy. I now choose units in which to measure leisure so that at this Pareto optimum, each consumer sells one unit of labor and consumes

$L - 1$ units of leisure. Next, I assume that during each trading period every consumer offers to sell exactly one unit of labor at the wage he quotes. Since the set of all consumers has measure one, it follows that the aggregate supply of labor during each trading period is unity. Thus, there is an institutionally defined "work day" in this economy; if a consumer works at all, he must work a full day.

Of course, a consumer could choose not to work at all. Since we assume that he always offers to sell one unit of labor, it must be that the consumption bundle he obtains when employed at the wage rate he quotes is, in fact, preferable to that obtained when not employed. Consequently, the assumption of an "inelastic" labor supply is actually an implicit restriction upon the set of observable wage densities. We now make this restriction precise.

Assume that at the start of each trading period the government writes each consumer a check for G "dollars" or units of account. Because the total "number" of consumers is unity, G is also the nominal value of the aggregate transfer from the government treasury to consumers. In this economy there is no opportunity for savings or investment. Therefore, if a worker is employed during a trading period at a nominal wage of w, his consumption bundle is $((G + w)/p, L - 1)$. Here p denotes the nominal price of a unit of output. If the worker is unemployed, his consumption bundle is $(G/p, L)$.

We now assume that if the nominal output price during a particular trading period is p, and the level of nominal government expenditures during that period is G, then any wage density x observed during that period satisfies

$$u((G + w)/p, L - 1) \geqslant u(G/p, L) \qquad \text{if } x(w) > 0.$$

The reader should note that this last assumption places a floor on the real wage during a trading period that depends upon p and G. However, in equilibrium, the average real wage will always *exceed* this floor. Consequently, this sort of wage "rigidity" plays no essential role in the theory.

I now turn to the second question posed above: How does a worker select the wage he quotes during a given trading period? Our approach will be to place certain restrictions upon this choice, but to allow a degree of indeterminateness within these limits.

In general, it will not be true that all workers are employed during every trading period. Let $z(w)$ denote the density of "hires" during a given period. Thus, $Z(w) = \int_0^w z(s)\,ds$ is the "number" of workers who were employed during the trading period in question. (The way in which $Z(w)$ is determined will be discussed in the following section.) Obviously $z(w) \leqslant x(w)$.

Now suppose that at the close of the trading period we are considering, each consumer can observe the employment rate at *every* wage w satisfying $x(w) > 0$. These numbers are just the ratios $z(w)/x(w)$, and might be published by the National Bureau of Labor Statistics. Each worker is assumed to interpret $z(w)/x(w)$ as the probability that he would have been employed during the period in question had he quoted a nominal wage of w. His ex post expected utility at w is then given by

$$U(w) = \frac{z(w)}{x(w)} u\left(\frac{G+w}{p}, L-1\right) + \left(1 - \frac{z(w)}{x(w)}\right) u\left(\frac{G}{p}, L\right).$$

Finally, we assume that if a worker quoted w during the period in question, he again quotes w during the following period *if and only if*

$$U(w) \geqslant U(w') \qquad \text{for all } w' \text{ satisfying } x(w') > 0.$$

If this condition is not met, the worker quotes some wage $\bar{w}$ that satisfies

$$U(\bar{w}) > U(w).$$

Thus, a worker continues to quote w if and only if w achieved the highest ex post expected utility among all wage rates actually quoted by consumers. If w does not meet this condition, then the consumer is assumed to select his next wage quotation so that it achieves a higher ex post expected utility than did w. Note, however, that this *change* in wage rates, though goal oriented, is not "optimal" in the standard sense.

I close this section by assuming that if the density of hires during a trading period is z, then the aggregate demand for output by consumers is

$$\frac{1}{p}\left(G + \int wz(w)\, dw\right).$$

Demand for output is "income constrained." Direct barter of labor for output is assumed to be impossible.

2. THE LABOR MARKET

How is the density of hires z determined during a typical trading period? We address this question in this section.

I shall first list the assumptions I wish to make about the process that determines the distribution of hires $Z(w)$. Once the list is complete, I shall describe a model of the labor market that satisfies these assumptions.

Denote the aggregate demand for labor by firms during the trading period in question by l_d. This is a *number*, not a function, and is assumed fixed throughout this discussion. The way in which l_d is determined by firms will be described in the following section.

L1. There exists a function $\theta(\alpha, \beta)$ such that

$$Z(w) = \theta(X(w), l_d).$$

θ is continuous; θ is smooth provided that $0 \leq \alpha \leq 1, 0 < \beta < 1$. Thus, to know the number of workers hired at wages not exceeding w we need only know the number of workers quoting wages not exceeding w and the aggregate demand for labor.

L2. $\lim_{w \to \infty} \theta(X(w), l_d) = \min(l_d, 1)$.

This states that the number of workers hired during a trading period equals either labor demand or labor supply, whichever is smaller.

L3. For $\beta > 0$,

(a) $D_1\theta(\alpha, \beta) > 0$,

(b) $D_1\theta(0, \beta) = 1$,

(c) $D_1\theta(\alpha, \beta)$ is a strictly decreasing function of α, provided that $\beta < 1$.

To interpret this last assumption, first note that, except for a set of wage rates of measure zero, we have the equality

$$z(w) = D_1\theta(X(w), l_d)\, x(w).$$

Thus, when $x(w) > 0$, the employment rate at w is just $D_1\theta(X(w), l_d)$. Assumption L3 now says that

(a) when labor demand is positive, the employment rate is positive for all w satisfying $x(w) > 0$;

(b) workers quoting the lowest wages are employed with probability one;

(c) the employment rate is a strictly decreasing function of the wage.

I shall now describe a model of the labor market that is consistent with these assumptions. To clarify ideas, I shall first describe a preliminary example where the number of firms and consumers is finite. The market consistent with L1–L3 will then be described by the analogous market mechanism, but with a continuum of both consumers and firms.

To describe the finite market, let n denote the number of consumers, and m the number of firms. Let $\lambda \equiv n/m$ be a large integer; λ will often be referred to as the size of a firm's labor pool. Let aggregate labor demand be an integer $l_d \leq n$, and suppose l_d is divisible by m, so that $l = l_d/m$ is each firm's demand for labor. All firms are assumed identical.

At the beginning of each trading period, every worker must find a job anew. Thus, each firm must hire its entire labor force, starting from scratch, in each period. The mechanism by which workers find jobs is described by the following postulates.

M1. Each firm hires workers from a pool consisting of precisely $\lambda = n/m$ individuals.

M2. The composition of a firm's labor pool is determined by a random mechanism described by the condition that every set of λ workers is equally likely to be a given firm's labor pool.

M3. Each firm hires l workers from its labor pool; workers are hired in order of ascending wage rates; thus the low-wage workers are hired before the high-wage workers in a firm's labor pool.

The important feature of this labor market is that, during any given trading period, there may be some high-wage workers who are employed and some low-wage workers who are not. A high-wage worker might be "lucky," and find himself in a labor pool with many workers quoting wage rates higher than his own. Similarly, a low-wage worker may be "unlucky," and find himself in a labor pool with many workers quoting wage rates lower than his own.

Now consider a market whose operation is described by postulates M1–M3, but in which there is a continuum of both firms and consumers. The measure of the set of all consumers is one, while the size of each firm's labor pool is a large integer λ. Thus, the measure of the set of all firms is $1/\lambda$. Let the aggregate demand for labor be l_d, and the demand for labor by a typical firm be $l_d/(1/\lambda) = \lambda l_d$.

I now assert that if one defines $Z(w)$ to be the expected value of the number of workers hired at wages not exceeding w in this infinite market, then $Z(w)$ satisfies L1–L3. This will be argued in some detail in Appendix 1.

The intuition underlying this result is straightforward. $Z(w)$ is the expectation of an infinite "sum" of random variables; each constituent of this sum is a random variable recording the number of workers hired at wage rates not exceeding w by a particular firm. Of course, these random variables are not independent. But since "expected value" is a linear operator, and since all firms are identical, the expected value of $Z(w)$ is completely determined by the expected value of any one of its constituent random variables.

We next apply the assumption that any given firm's labor pool is a set of measure zero in the population of all workers, i.e., λ is finite while the set of all workers is infinite. This allows us to treat a single firm's labor pool as being determined by λ *independent* drawings from the wage distribution X. The desired properties of $Z(w)$ follow immediately via an elementary calculation (see Appendix 1).

Before proceeding to the next section, I would like to make some technical remarks about this probabilistic model of the labor market.

First of all, the distribution of hires in a fixed trading period is, strictly speaking, a random variable. I have interpreted $Z(w)$ to be a "certainty equivalent" of this variable, obtained by taking expectations in a particular way. This might be technically justifiable by appealing to an appropriate law of large numbers; I do not know whether this is possible.

How would the economic model in this paper be altered if one recognized explicitly the random nature of the distribution of hires? It will become apparent in Section 4 that whether or not a wage distribution satisfies the "equal expected utility" condition, E2, depends only upon the expected value of the number of workers hired at each wage rate. Consequently, this aspect of the model does not depend upon whether one uses our certainty equivalent $Z(w)$ or instead allows the hire distribution to be explicitly random.

On the other hand, if the hire distribution is interpreted as random, then the aggregate demand for output must also be a random variable. Equilibrium condition E1 in Section 4 must then be replaced by the condition that the expected value of excess output demand be zero.

The second point involves assumption L3(a). This assumption implies that the employment rate for workers quoting the highest wages, $D_1\theta(1, \beta)$, be strictly positive. However, it will be apparent in Appendix 1 that the labor market model I have described results in $D_1\theta(1, \beta) = 0$. Our main theorem in Section 4 remains true if L3a is altered in this way, and is proved in exactly the same way. An additional assumption is needed to assure that the wage distribution has a finite mean in equilibrium.

3. FIRMS

I shall now discuss the way in which the aggregate demand for labor by firms is determined. The assumption is a simple one. Aggregate labor demand is determined by the condition that the average real wage of employed workers be equated with the marginal product of labor. A moment's thought makes clear that this "rule of thumb" is not a profit maximizing one. The consequences of "optimal" hiring behavior by firms are discussed later in this section and in Appendix 2.

I begin by assuming that all firms are identical in every respect; in particular, each demands the same quantity of labor as any other during a given trading period. In such circumstances it is harmless to assume the existence of an aggregate, short-run production function, $f(l)$. This is assumed to be strictly concave, for the usual reasons.

At the start of each trading period, I assume that all firms share a common point expectation, w_E, of the average nominal wage of employed workers for that period. An individual firm's demand for labor is determined by equating the marginal product of labor at that firm with w_E/p, the expectation the firm holds about the average real wage of employed workers. (Of course, all firms are assumed to know the nominal output price p, that will prevail during the period.) Finally, I suppose that the aggregate production function f is such that aggregate labor demand l_d satisfies the equation

$$Df(l_d) = w_E/p.$$

This then defines aggregate labor demand, $l_d(w_E/p)$, as a function of firms' expectations of the average real wage of employed workers.

I next would like to make some remarks about firms' profits.

Because the short-run production function is strictly concave, there will always be a positive level of aggregate profits in equilibrium. If the output market is to clear, these profits must somehow reenter the expenditure stream. In most general equilibrium models this is done by assuming that firms are owned by consumers and that all profits are paid out to them in the form of dividends.

In this model I take a different approach. I suppose that all firms are "owned" by the government; alternatively, one might assume that there is a 100% tax on corporate profits. Government revenue thus equals firm's profits, and reenters the expenditure stream via the "checks" the government writes each consumer at the beginning of each trading period. This approach is of interest because, by allowing the government to create and destroy "dollars" or units of account, the level of aggregate nominal government expenditure, G, can be regarded as an exogenous policy variable that affects the level of employment.

I close this section with some remarks about the consequences of "optimal" hiring policies by firms.

If a firm wishes to maximize its profits during a given trading period, its demand for labor will depend upon the composition of its labor pool during that period. The optimal hiring rule is to hire workers in order of increasing wage rates, and to stop when the next worker to be hired would be paid a real wage that exceeds his marginal product.

In Appendix 2 I argue that if all firms hire workers in this way, the labor market model described in the previous section retains its essential features. In particular, the expected number of workers hired at each real wage not exceeding $Df(0)$ is positive. The awkward consequence of this sort of hiring rule is that aggregate demand for labor and aggregate demand for output are, in principle, random variables. In particular, the random characteristics of output demand must be taken into account by any firm that claims to be "maximizing" its profits. I note in Section 2, however, that if these random variables can be regarded as concentrated on a single point, then with some additional assumptions, the equilibrium model in this paper can be reworked and leads to the same conclusions.

4. EQUILIBRIUM

I now define the appropriate concept of equilibrium for the economic system described in the previous sections. The main result of this section is that there is one such equilibrium corresponding to any given level of unemployment.

DEFINITION 4.1. An *equilibrium* is a 4-tuple p^*, $w_E{}^*$, X^*, G^* satisfying

E1: $f(l_d(w_E{}^*/p^*)) = (1/p^*)(G^* + \int wz^*(w)\,dw)$;

E2: for all w such that $x^*(w) > 0$,

$$D_1\theta\left(X^*(w), l_d\left(\frac{w_E{}^*}{p^*}\right)\right) u\left(\frac{w + G^*}{p^*}, L - 1\right)$$

$$+ \left(1 - D_1\theta\left(X^*(w), l_d\left(\frac{w_E{}^*}{p^*}\right)\right)\right) u\left(\frac{G^*}{p^*}, L\right) = t^*,$$

where t^* is a constant independent of w. Furthermore, $t^* > u(G^*/p^*, L)$.

E3: $w_E{}^* l_d(w_E{}^*/p^*) = \int wz^*(w)\,dw$.

Condition E1 states that the output market must clear in equilibrium. Condition E2 requires that all consumers achieve the same level of expected utility t^*, and that t^* exceeds the utility of the consumption bundle a worker could obtain if he chose not to work. Thus, in equilibrium, no worker perceives any advantage to changing the nominal wage rate he is quoting. Finally, E3 states that firms' point expectation of the average real wage of employed workers is confirmed. Notice that since all firms are treated symmetrically in the labor market, they all achieve the same level of expected profits.

Equilibria in this economy have two obvious properties.

First, aggregate demand for labor must always be less than aggregate supply. For if this were not so, every worker would be employed with probability one. Since there is an infinite number of wage rates being quoted in equilibrium, there are workers with high wage rates who achieve a higher expected utility than other workers with low wage rates. This violates condition E2.

Second, the government's budget is balanced in equilibrium, i.e., G^*/p^* equals the level of aggregate profits in the economy. This is just a restatement of conditions E1 and E3.

Our main theorem is essentially a converse to these two observations.

THEOREM 4.2. *Suppose u is continuous, quasi-concave, and strictly monotone increasing in its first argument. Let* p^*, $w_E{}^*$, G^* *satisfy*

1. $0 < l_d(w_E^*/p^*) < 1$;
2. $p^* f(l_d(w_E^*/p^*)) - w_E^* l_d(w_E^*/p^*) = G^*/p^*$.

Then there exists a wage distribution X^* *such that* p^*, w_E^*, X^*, G^* *satisfy* E1, E2, E3.

The proof of this theorem requires the following three lemmas. To state these we make some preliminary definitions.

Let $t_0 = u(G^*/p^*, L)$. This, of course, is the utility of the consumption bundle a worker could obtain if he chose to remain unemployed during a trading period. Next, define functions $w_{\min}(t)$, $w_{\max}(t)$ for $t \geqslant t_0$, by the equations

$$t = D_1\theta(0, l_d^*)\, u\left(\frac{w_{\min}(t) + G^*}{p^*}, L - 1\right) + (1 - D_1\theta(0, l_d^*))\, t_0,$$

$$t = D_1\theta(1, l_d^*)\, u\left(\frac{w_{\max}(t) + G^*}{p^*}, L - 1\right) + (1 - D_1\theta(1, l_d^*))\, t_0.$$

We adopt the notation l_d^* as shorthand for $l_d(w_E^*/p^*)$. These two functions are, respectively, the minimum and maximum wage rates quoted by workers when each worker achieves the same level of expected utility, t.

LEMMA 4.3. *Suppose* u *is continuous, quasi-concave, and strictly monotone increasing in its first argument. Then*

(a) *if* $t > t_0$, $w_{\max}(t) > w_{\min}(t)$;

(b) $w_{\max}(t_0) = w_{\min}(t_0) < w_E^*$;

(c) $\lim_{t \to +\infty} w_{\min}(t) > w_E^*$;

(d) $w_{\min}$, $w_{\max}$ *are continuous in* t.

Proof. (a) Consider the function $h(\alpha)$, $0 \leqslant \alpha \leqslant 1$, defined by the equation

$$D_1\theta(\alpha, l_d^*) = \frac{t - t_0}{u([h(\alpha) + G^*]/p^*, L - 1) - t_0}.$$

From the definitions of $w_{\min}(t)$, $w_{\max}(t)$, we note that $h(0) = w_{\min}(t)$ and $h(1) = w_{\max}(t)$. As α increases from zero to one, $D_1(\alpha, l_d^*)$ strictly decreases. To maintain the equality, $h(\alpha)$ must increase because u is strictly monotone increasing in its first argument. Thus, if $t > t_0$, $w_{\max}(t) > w_{\min}(t)$.

(b) Suppose $t = t_0$. By definition, we then have

$$u\left(\frac{w_{\min}(t_0) + G^*}{p^*}, L - 1\right) = t_0 = u\left(\frac{w_{\max}(t_0) + G^*}{p^*}, L - 1\right).$$

Since u is monotone in its first argument, it follows that $w_{\min}(t_0) = w_{\max}(t_0)$.

It remains to show that $w_{\min}(t_0) < w_E^*$. To follow the argument, the reader should refer to Fig. 1. This depicts the economy's production possibilities set, in per capita terms, as well as a typical consumer's indifference curves.

We begin by observing that in this economy, firms always operate "at a tangency position." In other words, $Df(l_d^*) = w_E^*/p^*$. Thus the straight line *BDE* has slope $-(w_E^*/p^*)$. Furthermore, point *E* on this line represents the consumption bundle $(G^*/p^*, L)$; this is just a restatement of Hypothesis 2 of Theorem 4.2.

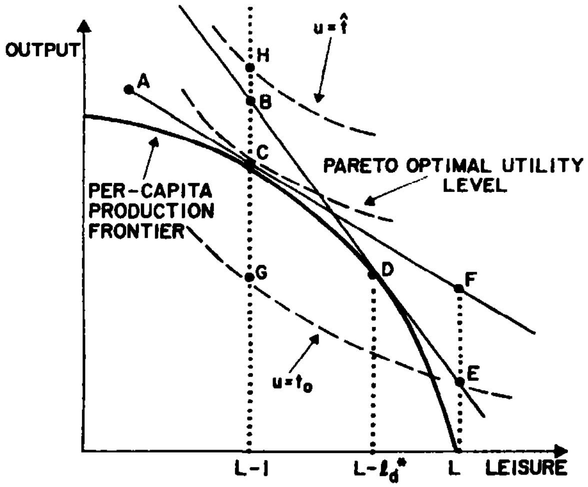

FIGURE 1

I assert that the utility of point *E* is less than the Pareto-optimal level of utility corresponding to the consumption bundle represented by point *C*. This follows from two observations. First, the slope of *BDE* $(= -(w_E^*/p^*))$ is algebraically less than the slope of ACF, for $L - l_d^*$ is greater than the Pareto-optimal level of leisure consumption $L - 1$; and so $-(w_E^*/p^*) = -Df(l_d^*) < -Df(1) =$ slope of *ACF*. In particular, point *E* lies southwest of *ACF*. The second observation is that, since preferences are convex, the set of all consumption bundles at least as good as point *C* lies northeast of *ACF*. Consequently, the utility of point *E* is less than the utility of point *C*.

Because u is increasing in its first argument, we conclude that the indifference curve passing through point *E* must intersect the vertical line with a leisure coordinate of $L - 1$ at point *G* lying below *C*.

Now, $w_{\min}(t_0)$ is defined to be that number such that $-(w_{\min}(t_0)/p^*)$ is the slope of the straight line joining points *G* and *E*. This is clearly algebraically greater than the slope of *BDE*, since *B* lies above *C*, and *C* lies above *G*. In other words, $w_{\min}(t_0) < w_E^*$.

(c) Let $\hat{t}$ be the utility index attached to an indifference curve intersecting the vertical line with leisure coordinate $L - 1$ at a point H, lying above point B. Then $w_{\min}(\hat{t})$ is defined by requiring $-(w_{\min}(\hat{t})/p^*)$ to be the slope of a straight line joining H to E. Since H lies above B, we conclude that $w_{\min}(\hat{t}) > w_E^*$ whenever $t \geq \hat{t}$.

(d) Continuity of $w_{\min}(t)$, $w_{\max}(t)$ follows from the continuity of u and its strict monotonicity in its first argument. Q.E.D.

LEMMA 4.4. *Suppose $t > t_0$. Then there exists a wage distribution X_t such that if x_t is the corresponding density, then*

(a) $x_t(w) = 0 \quad \text{if } w < w_{\min}(t) \quad \text{or} \quad w > w_{\max}(t)$;

(b) $$t = D_1\theta(X_t(w), l_d^*)\, u\left(\frac{w + G^*}{p^*}, L - 1\right) + (1 - D_1\theta(X_t(w), l_d^*))\, t_0$$

for all w satisfying $x_t(w) > 0$.

Proof. Fix $t > t_0$. Define a function $g(w)$ by the equation

$$g(w) = \frac{t - t_0}{u((w + G^*)/p^*, L - 1) - t_0}.$$

Recall that $D_1\theta$ is strictly decreasing in its first argument. Thus we conclude that when $w_{\min}(t) \leq w \leq w_{\max}(t)$, there is a unique real number $\nu(w)$ lying between zero and one that solves the equation

$$D_1\theta(\nu(w), l_d^*) = g(w).$$

Furthermore, $\nu(w_{\min}(t)) = 0$, $\nu(w_{\max}(t)) = 1$, $\nu(w)$ is continuous and increasing in w. Define

$$X_t(w) = \begin{cases} 0 & \text{for } w \leq w_{\min}(t), \\ \nu(w) & \text{for } w_{\min}(t) \leq w \leq w_{\max}(t), \\ 1 & \text{for } w \geq w_{\max}(t). \end{cases}$$

By construction, X_t satisfies the conclusions of the lemma. Q.E.D.

Before stating the next lemma, it is convenient to define, for $t > t_0$

$$\hat{w}_E(t) = \frac{1}{l_d^*} \int w D_1\theta(X_t(w), l_d^*)\, x_t(w)\, dw.$$

Thus, $\hat{w}_E(t)$ is the average wage of employed workers when labor demand is l_d^* and the wage distribution is X_t.

Lemma 4.5. *Let $t > t_0$. Then*

(a) *$\hat{w}_E(t)$ is continuous in t;*

(b) $\lim_{t \to t_0} \hat{w}_E(t) < w_E^*$;

(c) $\lim_{t \to \infty} \hat{w}_E(t) > w_E^*$.

Proof. (a) By construction, for fixed w, $X_t(w)$ varies continuously with t. If l_d^* is positive, then $D_1\theta(X_t(w), l_d^*)$ varies continuously with t. It follows that $\hat{w}_E(t)$ is continuous.

(b) Note that $\hat{w}_E(t) \leq w_{\max}(t)$. But, by Lemma 4.3,

$$\lim_{t \to t_0} w_{\max}(t) < w_E^*.$$

(c) Note that $\hat{w}_E(t) \geq w_{\min}(t)$. But, by Lemma 4.3,

$$\lim_{t \to +\infty} w_{\min}(t) > w_E^*. \qquad \text{Q.E.D.}$$

Proof of Theorem 4.2. From Lemma 4.5 we conclude that there is a t^* satisfying $\hat{w}_E(t^*) = w_E^*$. Define X^* to be X_{t^*}. Lemma 4.4 shows that X^* satisfies equilibrium condition E2. Our choice of t^* to satisfy $\hat{w}_E(t^*) = w_E^*$ shows that equilibrium condition E3 is satisfied. Finally, Hypothesis 2 of the theorem says that

$$f(l_d^*) = (1/p^*)[G^* + w_E^* l_d^*].$$

But our choice of t^* shows

$$w_E^* l_d^* = \int w z^*(w)\, dw.$$

Thus condition E1 is fulfilled. Consequently, p^*, w_E^*, X^*, G^* is an equilibrium. Q.E.D.

5. Concluding Remarks

Theorem 4.2 shows that, to each level of real government expenditure between zero and the level of aggregate profits at the Walrasian equilibrium, there corresponds an economic equilibrium with an aggregate excess supply of labor and a positive unemployment rate. Although the government's budget is balanced at each such equilibrium, the strict concavity of the short-run production function implies that the level of employment is positively related to the real level of government expenditures.

At each such equilibrium, every worker correctly perceives that he cannot increase his expected utility by lowering his wage quotation. For if he is quoting a wage above the minimum wage observed in equilibrium, lowering his wage quotation would increase the probability that he is employed but would also lower his wage income when employed. The net result is seen to be no increase in expected utility. Furthermore, if he happens to be quoting the lowest wage rate observed in equilibrium, he is employed with probability one. Again, there is no reason to lower his wage quotation.

This then is an example of an economy where some economic agents use quantity signals (the employment rates at various wages) as guides when setting prices. Because information about trading opportunities is imperfect (i.e., each firm hires from a labor pool that is extremely small relative to the size of the entire labor force), these quantity signals are not "Walrasian" ones. In other words, price setters do not find themselves to be perfect competitors. Consequently, there are economic equilibria where "rational" price setters see no incentive to change prices, yet collectively find themselves doing the "wrong" thing from society's point of view.

Indeed, I would like to argue that at least when workers are risk neutral, the unemployment appearing in this model is "involuntary" in the Keynesian sense. This is despite the fact that workers establish their wage rates voluntarily. I shall do this by showing that, when workers are risk neutral, the common level of expected utility achieved in equilibrium is a strictly increasing function of the aggregate equilibrium level of employment. Thus, each equilibrium is Pareto dominated (in expected utility terms) by any equilibrium with a higher aggregate level of employment. It follows from Theorem 4.2 that every equilibrium is Pareto dominated.

The crucial observation to make is that the level of expected utility achieved by risk neutral workers at an equilibrium equals the value of the utility function on the per-capita consumption bundle. Since indifference curves are convex, and since the "full employment" per-capita consumption bundle was defined so as to maximize the utility function over the per-capita production possibilities set, it then follows that the equilibrium level of expected utility is a strictly increasing function of the equilibrium level of employment.

I shall now establish the "crucial observation" mentioned above. Suppose workers are risk neutral. The indifference curves determined by u in output–leisure space are then straight lines. At equilibrium, all workers achieve the same level of expected utility. Risk neutrality then implies that the expected values of various workers consumption bundles all be along the same indifference curve. Since the indifference curve is a straight line, any convex combination of these expected values again lies on the same straight line. But the equilibrium per-capita consumption bundle is one such convex combination. Thus, the level of expected utility achieved by a worker at equilibrium equals the value of the utility function at the equilibrium per-capita consumption bundle.

We have therefore shown that, when workers are risk neutral, every equilibrium of this economy exhibits involuntary unemployment in the Keynesian sense.

It is tempting to believe that since higher levels of real government expenditure correspond to lower equilibrium unemployment rates, this economy can be effectively controlled by appropriate manipulation of this real expenditure level.

However, appealing this belief may be, it is important to note that, practically speaking, the variable most easily manipulated by policy makers is the level of *nominal*, not real, government expenditure. Changes in this nominal variable will, in general, result in wage and price level changes that tend to alter the level of employment in undesired ways. This is especially true if workers change their wage quotations in order to achieve higher expected utility.

Here is an illustrative example of such phenomena. Suppose initially that the economy is at an equilibrium with a positive unemployment rate. Let the government initiate a once and for all increase in the level of its nominal expenditure. The first effect of such action will be to to increase demand for output. Suppose that excess demand for output causes the nominal output price to rise, but that in the short run, the wage distribution remains fixed. Then the average real wage will begin to fall, the level of employment will begin to increase, and the supply of output will begin to increase. This process continues until the output market clears; this intermediate position has higher levels of employment, output, and real government expenditure, as well as a higher nominal output price than did the initial equilibrium.

So far, the initial wage distribution has been fixed. However, the demand for labor has increased; suppose this increase resulted in an increase in employment rates that was greater at high wages than at low wages. It will now be true that high-wage workers in this initial wage distribution achieve higher expected utility than do low-wage workers. Consequently, workers will begin to increase their wage quotations. It is not hard to see that if the average wage is rising, and the output market continues to clear, then the nominal output price must also rise. But then the real level of government expenditure begins to fall, as does the level of employment, while the real wage rises.

Notice that during this second stage of adjustment, the level of employment falls while the nominal average wage rises. Although all workers' "average" utility is falling, low-wage workers still see that it is possible to achieve higher expected utility by quoting higher wage rates.

Not only are the first- and second-stage effects of a change in nominal government expenditure upon the level of employment different, but assuming that the second stage terminates at a new equilibrium, just which equilibrium this will be is indeterminate without a detailed specification of the wage adjustment process. This problem arises because there is a continuum of possible equilibria.

As this example suggests, whether government policy can effectively control the level of economic activity in an economy where individual agents make "rational" price setting decisions is a subtle question. A persuasive answer must rest upon a detailed understanding of wage and price adjustment decisions that are made while the economy is not equilibrated. This problem awaits further analysis.

APPENDIX 1

In this Appendix I shall argue that when there is a continuum of firms and of consumers, the labor market model described in Section 2 fulfills assumptions L1–L3. Strictly speaking I shall show that this is true for a "certainty equivalent" of the probability model; in other words, if $Z(w)$ is defined to be the expected value of the number of workers hired, wage rates not exceeding w, then $Z(w)$ satisfies L1–L3.

The approach will be to first calculate, for any two wage rates $w'_1 > w'_0$, the expected number of workers, $Z(w'_1) - Z(w'_0)$, hired at wage rates between w'_0 and w'_1. We then show that the hire density,

$$z(w'_0) = \lim_{w'_1 \to w'_0} \frac{Z(w'_0) - Z(w'_0)}{w'_0 - w'_0},$$

can be written as $D_1\theta(X(w'_0), l_d)\, x(w'_0)$, where θ has the desired properties.

Before beginning, it should be noted that the same results hold for a labor market with only a countably infinite number of firms and of consumers. For such a market, it is possible to derive the desired results as characteristics of the limit of a sequence of markets with only finitely many participants.

Now assume there is a continuum of firms and a continuum of consumers. Let the measure of the set of all consumers be unity, and the measure of the set of all firms be $1/\lambda$, where λ is a large positive integer. Of course λ is the size of each firm's labor pool.

Denote the set of all firms by T, and let a typical firm be indexed by $t \in T$. For real numbers $w_1 > w_0$, denote the expected number of workers hired by firm t at wage rates between w_0 and w'_1 by $Z^t(w'_1) - Z^t(w_0)$. Because "expected value" is a linear operator, we know that

$$Z(w'_1) - Z(w_0) = \int_T (Z^t(w_1) - Z^t(w'_0))\, dt.$$

However, all firms are alike and treated symmetrically in the labor market. Consequently,

$$\int_T (Z^t(w'_1) - Z^t(w_0))\, dt = \frac{1}{\lambda}(Z^{t_0}(w'_1) - Z^{t_0}(w'_0))$$

for any $t_0 \in T$. We are now reduced to calculating the expected number of workers hired at wage rates between w_0 and w_1 by any single firm.

It is now important to notice that each firm's labor pool is finite, while the total population of consumers is infinite. Therefore, in order to calculate the relevant expectation, we may assume that the firm's labor pool is obtained by making λ *independent* drawings from the wage distribution X.

Let k denote the number of workers in a firm's labor pool quoting wage rates not exceeding w_0. Let j denote the number quoting wage rates between w_0 and w_1. Denote the firm's demand for labor by λl_d, where l_d is the aggregate demand for labor. Assume this quantity is integral. Then

$$\begin{aligned} Z(w_1) - Z(w_0) &= \frac{1}{\lambda}(Z'(w_1) - Z'(w_0)) \\ &= \frac{1}{\lambda} \sum_{k=0}^{\lambda l_d} \sum_{j=0}^{\lambda-k} \min(\lambda l_d - k, j)\binom{\lambda}{k}\binom{\lambda-k}{j} \\ &\quad \cdot X(w_0)^k (X(w_1) - X(w_0))^j (1 - X(w_1))^{\lambda-j-k}. \end{aligned}$$

Of course, $X(w_0)$ is the probability that the firm draws a worker quoting a wage rate less than w_0; $(X(w_1) - X(w_0))$ is the probability that the firm draws a worker quoting a wage between w_0 and w_1; $1 - X(w_1)$ is the probability that the firm draws a worker quoting a wage above w_1.

Now, for almost every w, the density of the expected number of hires at w_0 is given by

$$z(w_0) = \lim_{w_1 \to w_0} \frac{Z(w_1) - Z(w_0)}{w_1 - w_0}.$$

To calculate this limit, first observe that

$$\lim_{w_1 \to w_0} \frac{[X(w_1) - X(w_0)]^j}{w_1 - w_0} = \begin{cases} 0, & \text{if } j > 1, \\ x(w_0) & \text{if } j = 1. \end{cases}$$

Next observe that the terms in the above sum with index $j = 0$ or $k = \lambda l_d$ make a zero contribution to the sum. Consequently

$$\begin{aligned} z(w_0) &= \frac{1}{\lambda} x(w_0) \sum_{k=0}^{\lambda l_d - 1} \binom{\lambda}{k}(\lambda - k)\, X(w_0)^k (1 - X(w_0))^{\lambda-k-1} \\ &= x(w_0) \sum_{k=0}^{\lambda l_d - 1} \binom{\lambda - 1}{k} X(w_0)^k (1 - X(w_0))^{\lambda-k-1} \\ &\equiv x(w_0)\, D_1\theta(X(w_0), l_d). \end{aligned}$$

Clearly, $D_1\theta$ is a strictly decreasing function of its first argument. Furthermore, $D_1\theta > 0$ if $X(w_0) < 1$, and $D_1\theta = 1$ if $X(w_0) = 0$. Define

$$\theta(\alpha, l_d) = \int_0^{\alpha} D_1\theta(s, l_d)\, ds.$$

Then θ satisfies L1–L3, except for the requirement that $D_1\theta(1, l_d) > 0$. This failure was discussed at the end of Section 2.

APPENDIX 2

In this Appendix I shall discuss the consequences of assuming that each firm hires workers from its labor pool in each period so as to maximize its profits during that period.

The optimal hiring rule is to first order the workers in the labor pool according to increasing wage rates, and to hire workers in this order until the point where the next worker to be hired would be paid a real wage that exceeds his marginal product.

A labor market in which all firms use this hiring rule shares a very important characteristic with the labor market discussed in this paper. All workers quoting wage rates w that satisfy $X(w) < 1$ have a positive probability of being employed.

To show this, let f denote an *individual* firm's production function; assume $Df(0) = +\infty$. Define a function $l(w/P)$ by the equation

$$Df(l(w/P)) = w/P.$$

Thus, $l(w/P)$ is the maximum number of workers a firm could hire so that the marginal product of the last worker was at least w/P.

Consider a worker quoting a wage w with the property that $l(w/P)$ is an integer. The probability that this worker is hired equals the probability that he is hired by any given firm. This worker is hired by a firm if and only if the number of workers in that firm's labor pool quoting wage rates less than w is no greater than $l(w/P) - 1$. The probability of this event is

$$\sum_{k=1}^{l(w/P)-1} \binom{\lambda - 1}{k} X(w)^k (1 - X(w))^{\lambda - k - 1},$$

where λ is the size of a firm's labor pool. This derivation depends, of course, upon the fact that λ is finite, while the total number of workers is infinite; for then a firm's labor pool can be assumed to consist of λ independent drawings from the wage distribution X.

The next stage in the analysis of this kind of labor market is to ask whether, for a fixed p, there is a wage distribution X that causes each worker to achieve the same level of expected utility. The answer is yes, provided one makes some curvature assumptions upon the production function f. In particular, one must suppose that D^2f is large so that $Dl(w/P)$ will be small.

If this assumption is made, and if the randomness in aggregate output supply and demand can be ignored (perhaps via an application of an appropriate law of large numbers), then an economy with this sort of labor market can be shown to have equilibria with essentially arbitrary levels of employment. Of course, an equilibrium is a triple p^*, X^*, G^* such that the output market clears and all workers achieve the same level of expected utility.

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JOURNAL OF ECONOMIC THEORY 14, 221-222 (1977)

Notes, Comments, and Letters to the Editor

Editorial Addendum to "A Note on Rent and the Coase Theorem" by David Shapiro

Shapiro's note [1] on the Coase theorem has generated some confusion, and considerable correspondence to the Editors. In particular, in some correspondence (Brian Horrigan, UCLA, Alfred Endres, Universität Dortmund, and Mark Crain, Robert Tollison, and David Saurman, Texas A&M) it was argued that Shapiro was wrong in claiming a counterexample to the Coase Theorem. It appears to us that no one is really "wrong" in this case, but there is a problem of interpretation in the Shapiro example.

The example in question deals with the familiar farmer–cowman problem in which trampling herds are an external diseconomy to farmers. Shapiro points out that if there are sufficiently large set-up costs to the individual operations, the optimal solution will be to have only one activity in operation, and further that any proposal having both in operation at once will be unprofitable. Then he argues that whichever operation gets the "property right" initially will be the one in operation, violating the Coase proposition that the outcome will be independent of property right. This argument is correct as long as we restrict ourselves to *marginal* bargains. For example, if the farmer has a right to "untrampled fields," no cowman will be able to pay for the rights to graze one (or even three) cows and still turn a profit; the set up costs will be too large to cover variable cost. However, as the aforementioned correspondents point out, the Coase Theorem still holds if *large* bargains are allowed. Since the farmer and the cowman each make the same profits in isolation, the cowman will be able to compensate the farmer for getting out of his way entirely and still wind up with zero profits. Either outcome is possible, regardless of property rights, and the Coase Theorem stands.

Clearly, there are nonconvexities in the Shapiro example (set-up costs, at the very least). These nonconvexities explain how it is possible that marginal incentives do not lead to the global optimum. Looked at this way, the Shapiro article and the correspondence to follow were anticipated by a previous author in this Journal (D. Starrett) in his article, "Fundamental Nonconvexities in the Theory of Externalities" [2]. He showed that if there were full convexity in the externality generating environment,

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then there would always be sufficient "quasi-rent" so that the Shapiro example could not arise. However, he also showed that there were nonconvexities inherent in the problem of external diseconomies, nonconvexities which destroy the validity of the Coase Theorem. In particular, he showed that there might not be any bargaining equilibrium at all, and further that marginal incentive schemes such as the Pigovian tax system would lead to various different (not necessarily optimal) outcomes, depending on the assignment of property rights and the initial position.

THE EDITORS

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JOURNAL OF ECONOMIC THEORY 14, 223-227 (1977)

Closedness of Convex Hulls*

1. INTRODUCTION

As is well known, the convex hull of a closed set need not be closed, unless it is compact. This situation is readily illustrated in Fig. 1, in which the convex hull of the unbounded shaded part is not closed.

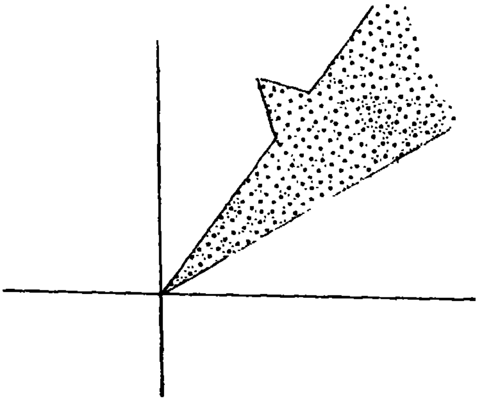

FIGURE 1

Thus it is often convenient, especially when one convexifies an economy with nonconvex consumption and production sets, to have a sufficient condition for the convex hull of a closed set to be closed. Starr [1, Appendix 3] showed it is sufficient for the closed set S to satisfy (A) S is a subset of R_+^n, the nonnegative orthant of the n-dimensional Euclidean space R^n, and (B) for any $y \in S$ there exist positive real numbers $M_1, \ldots, M_n$ such that $x^j = y + M_j e^j \in S$, $j = 1, \ldots, n$, where $e^j = (0, \ldots, 0, \underset{j}{1}, 0, \ldots, 0)$. While his result is useful for consumption sets, it is inapplicable to production sets. The purpose of this brief note is to give a more general condition in this line, with special regard to its application to production sets.

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2. NOTATION AND DEFINITIONS

We will use the following notation and definitions:

$|x|$ = Euclidean or any other equivalent norm of x in R^n,

con X = convex hull of a set X,

cl con X = closure of con X,

$P(X)$ = $\{y \in R^n: y = qx$ *for some real* $q \geq 0,\ x \in X\}$, i.e., the cone generated by X.

For *a* set S,

$S^v = \{x \in S: |x| \geq v\}$, where v is a nonnegative real number,

$G(S^v)$ = closure of $P(S^v)$,

$AS = \bigcap_{v>0} G(S^v)$, the asymptotic cone of S.

DEFINITION 1. A set X is *extendable* by a set A, if for each $x \in X$ and each $a \in A$, there exists $\epsilon > 0$, such that $x + \epsilon a \in X$.

DEFINITION 2. A set X *is spanned by* a set A, if every point of X is a nonnegative combination of points of A.

3. MAIN RESULTS

The following lemma plays a vital role in deriving the main results.

LEMMA. *If a closed set S is extendable by a set A, then*

$$\text{con } S + A \subset \text{con } S, \tag{1}$$

$$\text{con } S + P(A) \subset \text{con } S. \tag{2}$$

Further if $0 \in A$, *then*

$$\text{con } S + A = \text{con } S, \tag{1'}$$

$$\text{con } S + P(A) = \text{con } S. \tag{2'}$$

Proof. To prove (1) let $z \in \text{con } S + A$. Then

$$z = \sum p_i a^i + y, \qquad a^i \in S, \quad y \in A, \quad p_i \geq 0, \quad \sum p_i = 1,$$

so that by rearrangement,

$$z = \sum p_i(a^i + y).$$

Extendability means $x^i = a^i + M_i y \in S$ for positive M_i. Moreover, because of the closedness of S, we can take the above M_i arbitrarily large, so that we may assume $M_i \geqq 1$. Thus z can be represented as

$$\begin{aligned} z &= \sum p_i(a^i + (x^i - a^i)/M_i)) \\ &= \sum p_i\{(1 - (1/M_i))\, a^i + (1/M_i)\, x^i\} \in S, \end{aligned}$$

which proves (1).

It is readily seen that the extendability of S by A implies that by $P(A)$, so (2) reduces to (1). Equations (1′) and (2′) are obvious. Q.E.D.

The following corollary is immediate.

COROLLARY. *If a set B is spanned by a set A, by which a closed set S is extendable, then*

$$\operatorname{con} S + B \subset \operatorname{con} S.$$

PROPOSITION. *If a closed set S contains* 0, *then*

$$\operatorname{cl} \operatorname{con} S \subset \operatorname{con} S + \bigcap_{v \geqq 0} \operatorname{con} G(S^v). \tag{3}$$

Proof. Let $x \in \operatorname{cl} \operatorname{con} S$. There is a sequence $\{x(k)\}$ such that

$$\begin{gathered} x(k) \to x, \qquad x(k) = \sum p_i(k)\, a^i(k), \\ p_i(k) \geqq 0, \qquad \sum p_i(k) = 1, \qquad a^i(k) \in S. \end{gathered}$$

Taking a subsequence and relabeling, we may assume

$$\begin{aligned} &p_i(k) \to p_i, \\ &a^i(k) \to a^i \qquad (i \text{ in, say Class I}), \\ &|\, a_i(k)| \to +\infty \qquad (i \text{ in Class II}). \end{aligned}$$

For i in Class II and for any given nonnegative v, $a^i(k) \in S^v$ for sufficiently large k. Therefore, $p_i(k)\, a^i(k)$ belongs to $P(S^v)$, and hence is a member of $G(S^v)$. Because

$$x(k) = \sum_{\mathrm{I}} p_i(k)\, a^i(k) + \sum_{\mathrm{II}} p_i(k) \cdot 0 + \sum_{\mathrm{II}} p_i(k)\, a^i(k) \quad \text{and} \quad 0 \in G(S^v),$$

$$x \in \operatorname{con} S + \operatorname{con} G(S^v).$$

Arbitrariness of v implies

$$x \in \operatorname{con} S + \bigcap_{v \geqq 0} \operatorname{con} G(S^v),$$

which proves (3). Q.E.D.

Alternatively we can assume the boundedness of S from below (or above) instead of $0 \in S$ to prove (3) by essentially the same method. Further, we have

$$\bigcap_{v>0} \operatorname{con} G(S^v) = AS, \tag{4}$$

and hence

$$\operatorname{cl} \operatorname{con} S \subset \operatorname{con} S + AS,$$

if $G(S^v)$ is convex for every sufficiently large $v \geqq 0$.

In general, (4) does not hold. Even a less stringent relation,

$$\bigcap_{v>0} \operatorname{con} G(S^v) = \operatorname{con} AS$$

need not be true, which is evidenced by Fig. 2.

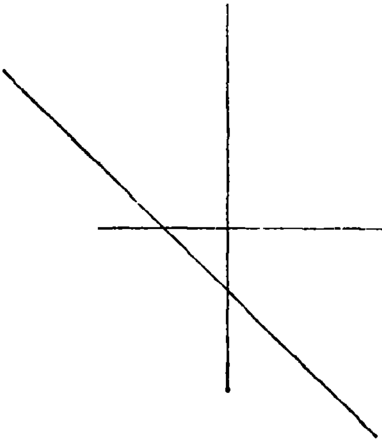

FIGURE 2

The discussions so far lead to the following theorem.

THEOREM. *Suppose that S is a closed subset of R^n, A is a subset of R^n, and S is extendable by A. If*

(a) *S contains 0 and A spans $\bigcap_{v>0} \operatorname{con} G(S^v)$, or*

(b) *S is bounded from below and A spans $\bigcap_{v>0} \operatorname{con} G(S^v)$, then* con *$S$ is closed.*

It is noted that (b) in the theorem generalizes Starr's condition.

It is also remarked that extendability in Definition 1 can be put in a weaker form without affecting the results above. It suffices to assume the extendability by A of a subset of X that spans X, rather than of X itself.

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An Empirical Implication of Auspitz–Lieben–Edgeworth–Pareto Complementarity

It is shown that if a differentiable demand function satisfying the budget identity is generated by a strongly concave twice-differentiable utility function such that all commodities are complements according to the Auspitz–Lieben Edgeworth Pareto definition, then all goods are normal and the demand for each commodity is a decreasing function of its own price.

1. INTRODUCTION

On the assumption that a consumer's preferences as between bundles $x = (x_1, x_2, \ldots, x_n)$ of n commodities can be represented by a twice-differentiable and strongly concave utility function u defined in a neighborhood of the interior of the positive orthant E_+^n of n-dimensional Euclidean space, Auspitz and Lieben [2, p. 482] defined commodities i and j to be *complementary* at x if $u_{ij}(x) = \partial^2 u(x)/\partial x_i \partial x_j > 0$ for $i \neq j$. This definition was adopted by Edgeworth [4, p. 117n] and Pareto [11, Chap. IV, and Appendix, Sects. 12-13, pp. 505-507]. It is well known that it was rejected by Allen [1, p. 171n], Hicks and Allen [7, p. 60n], and Hicks [6, pp. 42–45] on the ground that the condition $u_{ij}(x) > 0$ was not invariant with respect to monotone transformations of the utility function u. It has generally been concluded, as a result, that the criterion has no empirical content and that, accordingly, Pareto's discussion of complementarity was "inconsistent" [14, p. 385] or even "confused" [12, p. 1280].

The question nevertheless remains whether the postulate that preferences in the neighborhood of some point[1] can be represented by *some* strongly concave utility function for which all commodities are complementary (in the Auspitz–Lieben–Edgeworth–Pareto, or ALEP, sense) has any empirical implications.[2] In fact, Pareto [11, Chap. IV, Sect. 49] asserted

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[1] Allen [1] qualified his objections to the ALEP concept on the ground that they applied only locally. Samuelson [12] has also indicated reservations on this score. However, I shall here be concerned only with local results.

[2] It is curious that this question has not been raised before, since it is known (cf. Debreu [3]) that the hypothesis of *independent commodities* (i.e., $u_{ij}(x) = 0$ for $i \neq j$) carries definite implications concerning the shapes of indifference surfaces, and indeed has been known for a long time (cf. Pareto [10]) to imply (in conjunction with diminishing marginal utility, $u_{ii}(x) < 0$, for $i = 1, 2, \ldots, n$) that all goods are normal. In fact,

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that this was the case, namely that the supposition implied that all commodities are normal, and that demand for each commodity is a decreasing function of its own price.[3] (The second of course follows from the first, by the Slutsky equation.) Unfortunately Pareto did not provide a proof; the purpose of this note is to fill this gap.

2. THE EMPIRICAL IMPLICATION

Let $p = (p_1, p_2, \ldots, p_n)$ be the price vector and let I denote income. Ω will denote the set of all price–income pairs (p, I) such that $p_i > 0$, $i = 1, 2, \ldots, n$, and $I > 0$. The individual's demand function will be denoted $h: \Omega \rightarrow E_+^n$, and its range denoted $X = \{x: x = h(p, I)$ for some $(p, I) \in \Omega\}$.

THEOREM. *Let an individual's preferences be such as to generate a single-valued, differentiable demand function* $h(p, I)$ *defined on* Ω *and satisfying the budget identity*

$$p \cdot h(p, I) = \sum_{i=1}^{n} p_i h^i(p, I) = I \quad \textit{for all} \quad (p, I) \in \Omega. \tag{1}$$

Let Ω^0 be a neighborhood whose image $X^0 = h(\Omega^0)$ is in the interior of E_+^n. Then if the individual's preferences over X^0 are representable by a real-valued, twice-differentiable utility function $u: X^0 \rightarrow E^1$ satisfying

(i) *strong concavity*, i.e., for all $x \in X^0$, the rth-order principal minors of the matrix $U(x) = [u_{ij}(x)] = [\partial^2 u(x)/\partial x_i \partial x_j]$ have the sign of $(-1)^r$;

(ii) *weak* ALEP *complementarity*, i.e., $u_{ij}(x) \geq 0$ for all $i \neq j$ and $x \in X^0$;

it follows that $h_I^i(p, I) = \partial h^i(p, I)/\partial I > 0$ for $i = 1, 2, \ldots, n$, and $(p, I) \in \Omega^0$.

Georgescu-Roegen [5] has shown that ALEP complementarity also influences the shapes of indifference curves, hence must have *some* empirical implications. It remains only to determine what these are.

[3] Edgeworth [4, 117n] showed that for the case $n = 2$, if the marginal utility of income was assumed to be independent of prices, a rise in the price of one commodity would lead to a fall in the demand for the other if the commodities were ALEP complements and a rise in the demand for the other if they were ALEP subsitutes. Generalizations of this result were obtained by Hotelling [8] and Schultz [13].

Proof. Denoting the indirect utility function by $v(p, I) = u[h(p, I)]$, it is well known that under the stipulated conditions we have

$$v_I(p, I) = u_i[h(p, I)]/p_i \qquad ((p, I) \in \Omega^0, \quad i = 1, 2, \dots, n), \tag{2}$$

where $v_I = \partial v/\partial I$. Differentiating (2) with respect to I and denoting $v_{II} = \partial^2 v/\partial I^2$, $h_I^i = \partial h^i/\partial I$, and denoting the column vectors $h_I = (h_I^i)$, $p = (p_i)$, we have

$$U[h(p, I)]\, h_I(p, I) = v_{II}(p, I)\, p \qquad \text{for} \quad (p, I) \in \Omega^0. \tag{3}$$

From assumptions (i) and (ii) it follows (see, e.g., McKenzie [9]) that the matrix $U(x)$ has a dominant negative diagonal, hence its inverse has only nonpositive entries, with at least one negative entry in each row. It follows that the components of the column vector

$$\{U[h(p, I)]\}^{-1} p$$

are all negative. Consequently it is clear from (3) that the components of $h_I(p, I)$ must all have the same sign, opposite to that of $v_{II}(p, I)$. If $v_{II}(p, I) \geqq 0$ then $h_I^i(p, I) \leqq 0$ for $i = 1, 2, \dots, n$, in violation of the budget constraint (1); therefore we must have $v_{II}(p, I) < 0$ for all $(p, I) \in \Omega^0$, and the result follows.

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The Definition of Risk: An Extension*

1. In a recent paper, Rothschild and Stiglitz [4] (hereafter, *R & S*) provide three definitions of risk for random variables with identical means. Since they show that these three indices coincide, we will discuss below only one measure that they suggest. Denoting the cumulative distributions of two risky options with *equal means* by F and G, *R & S* show that option G is riskier than option F if

$$\int_{-\infty}^{x} [G(t) - F(t)]\, dt \geq 0, \qquad \text{for all values } x. \tag{1}$$

The purpose of this note is to extend the above definition and to show that it is applicable to any two options without any constraint on the means, provided that one allows lending and borrowing at some riskless interest rate, r.

2. Let F and G be two risky options with means $E_F(x)$ and $E_G(x)$, respectively. Without loss of generality, assume that $E_G(x) > E_F(x)$. Borrow money at r and create from F another levered portfolio F_α, where $x_\alpha = \alpha x + (1 - \alpha) r$. Determine α to fulfill the constraint $E_{F_\alpha}(x) = E_G(x)$. Then $\alpha E_F(x) + (1 - \alpha) r = E_G(x)$ and $\alpha = (E_G(x) - r)/(E_F(x) - r)$. Clearly, if $E_G(x) > E_F(x)$ (as we assumed), $\alpha > 1$; thus, in this specific case F_α is a levered portfolio.

THEOREM. *Let F and G be two options with expected means $E_F(x)$ and $E_G(x)$, respectively. If there is one combination of F with a riskless asset, F_α, where $E_{F_\alpha}(x) = E_G(x)$, such that G is riskier than F_α, then for any other two combinations, G_β, F_δ, with $E_{G_\beta}(x) = E_{F_\delta}(x)$, G_β is riskier than F_δ. Thus, it is always possible to obtain a less risky position by diversifying F with riskless assets. Hence, F is a less risky option in spite of the fact that $E_F(x) \neq E_G(x)$.*

Proof. By assumption $E_{G_\beta}(x) = E_{F_\delta}(x)$, hence $\beta(E_G(x) - r) = \delta(E_F(x) - r)$, or

$$\delta = \beta[E_G(x) - r]/[E_F(x) - r] = \beta\alpha. \tag{2}$$

Given that G is riskier than F_α, we obtain

$$\int_{-\infty}^{x} [G(t) - F_\alpha(t)]\, dt \geq 0, \qquad \text{for all values } x. \tag{3}$$

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However, F_α can be rewritten as

$$F_\alpha(x) = Pr(x_\alpha \leq x) = Pr(\alpha x + (1-\alpha)r \leq x) = F((x-(1-\alpha)r)/\alpha).$$

Substituting for $F_\alpha(t)$ in (3), we obtain

$$\int_{-\infty}^{x}\left[G(t) - F\left(\frac{t-(1-\alpha)r}{\alpha}\right)\right]dt \geq 0, \quad \text{for all values } x. \tag{4}$$

We must prove that (4) implies

$$\int_{-\infty}^{x}[G_\beta(t) - F_\delta(t)]\,dt \geq 0, \quad \text{for all values } x. \tag{5}$$

But since $G_\beta(x) = G((x-(1-\beta)r)/\beta)$ and $F_\delta(x) = F((x-(1-\delta)r)/\delta)$, (5) can be rewritten as

$$\int_{-\infty}^{x}\left[G\left(\frac{t-(1-\beta)r}{\beta}\right) - F\left(\frac{t-(1-\delta)r}{\delta}\right)\right]dt \geq 0, \quad \text{for all values } x. \tag{6}$$

Recall that by (2), $\delta = \beta\alpha$. Thus, (6) can be rewritten as

$$\int_{-\infty}^{x}\left[G\left(\frac{t-(1-\beta)r}{\beta}\right) - F\left(\frac{t-(1-\alpha\beta)r}{\alpha\beta}\right)\right]dt$$

$$\int_{-\infty}^{x}\left[G\left(\frac{t-(1-\beta)r}{\beta}\right) - F\left(\frac{[t-(1-\beta)r]/\beta - (1-\alpha)r}{\alpha}\right)\right]dt \geq 0 \tag{7}$$

for all values of x. Thus, it is enough to show that (3) implies (7). But this is straightforward; in (7) make the transformation $u = (t-(1-\beta)r)/\beta$ to obtain

$$\beta\int_{-\infty}^{(x-(1-\beta)r)/\beta}\left[G(u) - F\left(\frac{u-(1-\alpha)r}{\alpha}\right)\right]du \geq 0. \tag{7'}$$

But since $\beta > 0$, (3) implies (7′) (or 7). Q.E.D.

Thus all one must do is equate the two means of the two random variables at any arbitrary level and then apply the risk measure suggested by Rothschild and Stiglitz. If, according to the chosen mean level one option is found to be riskier than the other, it will remain riskier for any other level of means that may be chosen.

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A Note on the Arrow–Kurz Model of Public Investment

The pioneering book by Arrow and Kurz [1] pertaining to public investment and optimal fiscal policy rigorously sets forth the foundation for many of the recent dynamic analyses in the area of public and natural resource economics.

The purpose of this note is to correct an error in the Arrow–Kurz (A–K) model of optimal public investment [1, Chap. 4]. The error appears in Proposition 2. This proposition states that if consumption and government capital are complements in social welfare, then along with other sufficient conditions, there exists a unique steady state which is a saddle point. We shall prove that, given the structure of the A–K model, there exists a unique saddle point steady state irrespective of the substitutability or complementarity of consumption and public capital.

The optimality model is

$$\max_{(c,k_g,k_p)} \int_0^\infty e^{-\lambda t} U(c(t), k_g(t))\, dt, \tag{1}$$

subject to

$$k_g(t) + k_p(t) = k(t),$$

$$\dot{k}(t) = f(k_p(t), k_g(t)) - \gamma k(t) - c(t),$$

where

c = consumption,
k_g = public capital,
k_p = private capital,
k = total capital

(all of the above variables are evaluated in efficient labor endowment terms),

U = the social welfare function which is strictly concave, homogeneous of degree $1 - \sigma$, $\sigma > 0$,
f = the production function which is strictly concave,
t = time,
$\lambda = \omega - \gamma$, where $\omega = \rho + \sigma\tau$ with ρ the social rate of discount and τ the rate of labor saving technological change; $\gamma = \pi + \tau$, where π is the rate of growth of the population.

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From (1) the optimality conditions are (we delete the time variable),

$$U_c = p, \tag{2}$$

$$U_g + p(f_g - f_p) = 0, \tag{3}$$

$$k_p + k_g = k, \tag{4}$$

$$\dot{k} = f(k_p, k_g) - \gamma k - c, \tag{5}$$

$$\dot{p}/p = \omega - f_p \tag{6}$$

Equations (2), (3), and (4) define the short-run equilibrium with the derived demand functions, $c = c(p, k)$, $k_g = k_g(p, k)$, and $k_p = k_p(p, k)$. Thus the steady state is derived by solving the system,

$$\begin{aligned} \Psi(p, k) &= f(k_p(p, k), k_g(p, k)) - \gamma k - c, \\ \phi(p, k) &= \omega - f_p(k_p(p, k), k_g(p, k)). \end{aligned} \tag{7}$$

Arrow and Kurz prove that $\partial\phi/\partial p > 0$, $\partial\phi/\partial k > 0$, and $\partial\Psi/\partial p > 0$. In addition they conclude that since

$$\frac{\partial\Psi}{\partial k} = f_p - \gamma + (f_g - f_p)\frac{\partial k_g}{\partial k} - \frac{\partial c}{\partial k}, \tag{8}$$

and so if $U_{cg} > 0$ then $\partial c/\partial k > 0$ and because $\partial k_g/\partial k > 0$ then the sign of $\partial\Psi/\partial k$ depends on $f_p - \gamma$. Moreover, when $\phi(p, k) = 0$ then $\omega = f_p$ and so if $\omega - \gamma$ is sufficiently small when $U_{cg} \geqslant 0$ (i.e., consumption and public capital are weak complements in social welfare), we find that $\partial\Psi/\partial k < 0$. This means that there exists a unique steady state which is a saddle point [1, Proposition 2, Chap. 4].

However, let us investigate the terms $\partial k_g/\partial k$ and $\partial c/\partial k$. From the optimality conditions,

$$\partial k_g/\partial k = (-U_{cc}/\Delta)\, p(f_{gp} - f_{pp}) > 0, \tag{9}$$

since $f_{gp} - f_{pp} > 0$ by assumption, and $\Delta > 0$ (where Δ is the relevant determinant), and

$$\partial c/\partial k = (U_{cg}/\Delta)\, p(f_{gp} - f_{pp}). \tag{10}$$

Therefore (8) becomes

$$\frac{\partial\Psi}{\partial k} = f_p - \gamma + (f_p - f_g)\frac{U_{cc}}{\Delta} p(f_{gp} - f_{pp}) - \frac{U_{cg}}{\Delta} p(f_{gp} - f_{pp}). \tag{11}$$

Utilizing Eqs. (2) and (3), we find that

$$\frac{\partial\Psi}{\partial k} = f_p - \gamma + \frac{U_g}{\Delta} p(f_{gp} - f_{pp}) \left[\frac{U_{cc}}{U_c} - \frac{U_{gc}}{U_g}\right]. \tag{12}$$

Because A–K also assumed that U is homogeneous of degree $1 - \sigma$, $\sigma > 0$, this implies that

$$(U_{cc}/U_c) - (U_{gc}/U_g) < 0.$$

Therefore, $(\partial\Psi/\partial k) - f_p + \gamma < 0$. Consequently, for the existence of a unique, saddle point steady state it must be the case that $\omega \quad \gamma$ is sufficiently small, irrespective of the weak substitutability or complementarity of consumption and government capital in social welfare.

In conclusion, the sufficient condition $U_{cg} > 0$ is redundant in the Arrow–Kurz model of optimal public investment.

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Externalities and Problems of Nonconvexity and Overhead Costs in Welfare Economics

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INTRODUCTION

A large part of the existing theory of welfare economics is based upon the relationships between competitive equilibrium and Pareto optimality which have been shown to exist in general equilibrium models of the Arrow–Debreu type. However, it is known that these relationships fail in models which, while retaining all other essential features of the Arrow–Debreu models, allow for the existence of nonmarket interrelationships, or "externalities" between agents. As a result, considerable effort has been expended in the attempt to develop alternative mechanisms for the efficient allocation of resources in an economy with externalities. Most suggested mechanisms maintain the basic framework of the competitive mechanism and introduce elements of centralized decision making which are designed to incorporate the costs and benefits involved with externalities into the market system.

Modifications of the competitive mechanism which are designed to deal with externalities are often classified into two general types; the first type includes schemes of taxes and subsidies in the Pigouvian tradition, and the second type includes systems of artificial markets on which externality rights are traded. Stimulated by the study of Coase [3], Negishi [10] and Starrett [13] have recently established, in a general equilibrium context, that all such schemes are based upon the same price characterization of efficient allocations, and in doing so these authors have provided a framework within which an analysis and comparison of the various schemes can be meaningfully conducted. However. Starrett casts doubt upon the appropriateness of this framework for dealing with an economy in which there are detrimental externalities. He argues that the convexity assumptions, which play a crucial role in establishing the existence of the price characterization, are inconsistent with the existence of detrimental externalities.

The present paper is intended to be primarily a complement to Starrett's work [13]. In Section 2 an alternative proof for the existence of the relevant price characterization is provided. This proof, which is based upon a theorem established by Rockafeller in the context of his work in the theory of duality

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in convex analysis [12], allows for a simple extention of Starrett's "equivalence theorem" to an economy in which there may be external effects among all agents and not only among producers. The methods employed are among those of the modern theory of competitive equilibrium, and it is hoped that the analysis presented in Section 2 will help to illustrate the implications which the consideration of externalities has for that theory. In Section 3 the question of the inconsistency of convexity assumptions in a model in which there are detrimental externalities is investigated in detail. The basic model and notation are presented in Section 1.

1. DEFINITIONS AND NOTATION

We consider an economy in which $L = \{1, 2,..., l\}$ is the set of commodities, $M = \{1, 2,..., m\}$ is the set of consumers, and $N = \{1, 2,..., n\}$ is the set of producers. The symbols j and k will be reserved to refer to commodities, h and i to consumers, and f and g to producers. A consumption allocation is an element $x = (x_1 ,..., x_h ,..., x_m)$ of the ml-dimensional Euclidean space R^{ml}. A production allocation is an element $(v, z) = (v_1 ,..., v_f ,..., v_n ; z_1 ,..., z_f ,..., z_n)$ of R_+^{2nl}, where R_+^{2nl} is the nonnegative orthant of R^{2nl}.

While it is customary in static equilibrium analysis to represent production allocations as vectors of net outputs, this convenience is less appropriate when externalities are introduced,[1] and we choose to represent the inputs and outputs of firm f by the nonnegative vectors v_f and z_f, respectively. A state of the economy is an element $a = (x, v, z)$ of $R^{m} \times R_+^{2nl}$. $R^{m} \times R_+^{2nl}$ will be alternatively denoted by D.

The consumption set of consumer i is a subset, C_i of D, on which there is defined a total, reflexive, and transitive preference ordering $\gtrsim_i$. $C = \bigcap_{i \in M} C_i$ is the feasible consumption set. The technology set of producer f is a subset, S_f, of D, and $S = \bigcap_{f \in N} S_f$ is the feasible technology set.

The following notation will be useful:

$$\begin{aligned}
G_i(a) &= \{a' \in C_i \mid a' \gtrsim_i a\},\\
P_i(a) &= \{a' \in C_i \mid a' \gtrsim_i a \text{ and not } a \gtrsim_i a'\},\\
G(a) &= \bigcap_{i \in M} G_i(a),\\
P(a, h) &= \bigcap_{\substack{i \in M\\ i \neq h}} G_i(a) \cap P_h(a),
\end{aligned}$$

$$\begin{aligned}
&\text{for } x = (x_1 ,..., x_m) \in R^{ml}, && x(M) = x_1 + \cdots + x_m ,\\
&\text{for } v = (v_1 ,..., v_n) \in R^{nl}, && v(N) = v_1 + \cdots + v_n ,\\
&\text{for } z = (z_1 ,..., z_n) \in R^{nl}, && z(N) = z_1 + \cdots + z_n .
\end{aligned}$$

---

[1] Certain reasons for this will become apparent in Section 3. For a more general discussion, the reader is referred to the survey article by Millerson [8, p. 424].

There is a vector of initial endowments ω, which is an element of the nonnegative orthant of R^l. Letting $H = \{a = (x, v, z) \in D \mid x(M) + v(N) - z(M) = \omega\}$, we define the set of attainable states to be $A = C \cap S \cap H$.

A state $a^* \in D$ will be called *Pareto efficient* if (i) $a^* \in A$, and (ii) for each $h \in M$, $P(a^*, h) \cap A = \emptyset$.

In Section 2, μ, s^h, and t^f will be elements of $R^{(m+2n)l}$ and will be referred to as price systems. In this case, we write, for example, $s^h = (s_x{}^h, s_v{}^h, s_z{}^h)$, where

$$s_x{}^h = (s_{x1}^h, \ldots, s_{xm}^h) \in R^{ml}, \qquad s_v{}^h = (s_{v1}^h, \ldots, s_{vn}^h) \in R^{nl},$$

and

$$s_z{}^h = (s_{z1}^h, \ldots, s_{zn}^h) \in R^{nl}.$$

s_{xi}^h may be thought of as a vector of l prices which consumer h is to pay or receive for the consumption of consumer i, and s_{vf}^h and s_{zf}^h as the prices which consumer h is to pay or receive for the inputs and outputs, respectively, of producer f. Similarly, t^f will be a vector of $(m + 2n)l$ prices which producer f pays or receives for the various agents' production and consumption.

If $a = (x, v, z) \in D$ and $v_f' \in R^l$, let $a(v_f') = (x, v', z)$, where $v' = (v_1, \ldots, v_f', \ldots, v_n)$. Similarly, if $v_{fj}' \in R^1$, let $a(v_{fj}')$ be the state obtained from a by replacing the component of a which represents the fth producer's input of good j, with v_{fj}'. $a(x_h)$, $a(x_{hi})$, $a(z_f)$, and $a(z_{fj})$ are defined analogously. Then for $(v_f, z_f) \in R^l \times R^l$, let $a((v_f, z_f)) = a(v_f)(z_f)$. Finally, let $S_f(a) = \{(v_f, z_f) \mid a((v_f, z_f)) \in S_f\}$.

2. A SHADOW PRICE CHARACTERIZATION OF EFFICIENCY

The purpose of this section is to present a shadow price characterization of the Pareto-efficient states of the economy described above. It is this characterization which is the basis for discussion of tax-subsidy and artificial market schemes for dealing with externalities. For this section only, we assume:

(A.1) *For each producer f, S^f is convex.*

THEOREM 1. *If a^* is a Pareto-efficient state such that*

(A.2) *for each consumer h, $P_h(a^*)$ is convex,*

(A.3) *for each consumer h, the closure of $P_h(a^*) = G_h(a^*)$,*

(A.4)[2] $\{\bigcap_{i \in M} \mathrm{ri}G_i(a^*)\} \cap \{\bigcap_{f \in N} \mathrm{ri}S_f\} \neq \emptyset$,

---

[2] For a convex set C, ri$C$ will denote the interior of C relative to the smallest affine space containing C.

then there exist a nonzero social price vector $p \in R^l$, a price system s^h for each consumer h, and a price system t^f for each producer, such that

(i) *for each $h \in M$, $a \in G_h(a^*)$ implies that $s^h \cdot a^* \leq s^h \cdot a$,*

(ii) *for each $f \in N$, $a \in S_f$ implies that $t^f \cdot a \leq t^f \cdot a^*$,*

(iii) *for each $i \in M$, $\sum_{h=1}^{m} s^h_{xi} = p + \sum_{f=1}^{n} t^f_{xi}$,*

(iv) *for each $f \in N$, $\sum_{h=1}^{m} s^h_{yf} = p + \sum_{g=1}^{n} t^g_{yf}$,*

(v) *for each $f \in N$, $\sum_{g=1}^{n} t^g_{zf} = p + \sum_{h=1}^{m} s^h_{zf}$.*

The purpose of assumptions (A.2) and (A.3) is to guarantee that

(2.1) for each consumer h, $G_h(a^*)$ is convex, and

(2.2) $a^* \notin \mathrm{ri}G(a^*)$.

By introducing the concept of satiation, (2.1) and (2.2) can be obtained from somewhat weaker assumptions. But since this procedure is well known (see, for example, [5, pp. 95–96]), we have chosen to simplify the exposition by making assumptions which yield (2.1) and (2.2) directly. Assumption (A.1) is discussed in detail in Section 3. Assumption (A.4) is a form of the familiar Slater constraint qualification, and examples which show that the theorem fails without this condition are easy to construct.

The proof of the theorem is based on the following lemma, which we quote directly from [12, p. 223], where its proof can also be found.

LEMMA. "*Let $C_1, \ldots, C_m$ be convex sets in R^n whose relative interiors have a point in common. Then the normal cone*[3] *to $C_1 \cap \cdots \cap C_m$ at any given point x is $K_1 + \cdots + K_m$*[4], *where K_i is the normal cone to C_i at x.*"

To prove Theorem 1, first note that $H = \{x + \omega \mid x \in L^\perp\}$, where $L^\perp$ is the orthogonal complement of the l-dimensional subspace, L, of $R^{(m+2n)l}$ defined $L = \{\mu = (\mu_{x1}, \ldots, \mu_{xm}; \mu_{y1}, \ldots, \mu_{yn}; \mu_{z1}, \ldots, \mu_{zn}) \mid -\mu_{x1} = \cdots = -\mu_{xm} = \mu_{y1} = \cdots = \mu_{yn} = \mu_{z1} = \cdots = \mu_{zn}\}$.

Define $T = G(a^*) \cap S$. Then, by (2.1) and (A.1), T is convex. Since a^* is attainable, $a^* \in T \cap H$. Condition (2.1) implies that $a^* \notin (\mathrm{ri}T \cap \mathrm{ri}H)$, since $H = \mathrm{ri}H$. It then follows, by a well-known separation theorem (see, for example, [12, p. 97]), that there is a hyperplane separating T and H at a^*. This hyperplane supports the affine subspace H, and hence must have a normal $\mu = (-p,\ldots, -p; -p,\ldots, -p; p,\ldots, p) \in L$. Then, since μ is normal to $T = \{\bigcap_{h\in m} G_h(a^*)\} \cap \{\bigcap_{f\in N} S_f\}$ at a^*, the conclusions of the theorem follow directly from the lemma.

---

[3] The normal cone to a convex set C at $x_0 \in C$ is the set $\{p \mid (\forall x \in C) p \cdot x \leq p \cdot x_0\}$.

[4] $K_1 + \cdots + K_m = \{k_1 + \cdots + k_m \mid k_i \in K_i, i = 1, \ldots, m\}$.

Conclusions (i)–(v) of Theorem 1 leave unanswered questions as to whether or not the characterizing prices correspond to the economist's intuition concerning artificial markets and tax subsidy prices. For example, if the fth producer's output of the ith commodity has a detrimental external effect on the technological possibilities of the gth producer, then we would expect that $t^g_{fzj} > 0$. In general, such intuitive expectations can be verified quite simply from properties (i)–(v), provided that a suitable definition of externality can be agreed upon. Instead of presenting a long list of such results, we choose one representative which will both, indicate the method by which the others can be derived and allow us to compare certain consequences of Theorem 1 with the results of Murakami and Negishi in [9].

The case which we consider is that under which the inputs or outputs of a given producer do not have external effects on the possibilities of the remainder of the economy taken as a whole. To be more precise, we let $T_f = \{\bigcap_{g\neq f} S_g\} \cap G(a^*)$, $T_h = \{\bigcap_{g\in N} S_g\} \cap \{\bigcap_{i\neq h} G_i(a^*)\}$, and say that

(a) the output (input) of the jth commodity at the fth producer has a *general detrimental externality* at a^* if $z_{fj} > z^*_{fj}$ ($v_{fj} > v^*_{fj}$) implies that $a^*(z_{fj}) \notin T_f(a^*(v_{fj}) \notin T_f)$,

(b) the output (input) of the jth commodity at the fth producer has a *general beneficial externality* at a^* if $z^*_{fj} \neq 0$ and $z_{fj} < z^*_{fj}$ ($v^*_{fj} \neq 0$ and $v_{fj} < v^*_{fj}$) imply that $a^*(z_{fj}) \notin T_f(a^*(v_{fj}) \notin T_f)$,

(c) the consumption of the jth commodity by the hth consumer has a *general detrimental* (*beneficial*) *externality* at a^* if $x_{hj} > x^*_{hj}$ ($x_{hj} < x^*_{hj}$) implies that $a^*(x_{hj}) \notin T_h$.

We may now establish the following results.

THEOREM 2. (i) *If the output* (*input*) *of the jth commodity at the fth producer does not have a general detrimental externality, then*

(i.1) *for each producer* $g \neq f$, $t^g_{zfj} \leqslant 0$ ($t^g_{vfj} \leqslant 0$),

(i.2) *for each consumer* h, $s^h_{zfj} \geqslant 0$ ($s^h_{vfj} \geqslant 0$), *and*

(i.3) $p_j \leqslant t^f_{zfj}$ ($-p_j \leqslant t^f_{vfj}$).

(ii) *If the output* (*input*) *of the jth commodity at the fth producer does not have a general beneficial externality, and if* $z^*_{fj} \neq 0$ ($v^*_{fj} \neq 0$), *then*

(ii.1) *for each producer* $g \neq f$, $t^g_{zfj} \geqslant 0$ ($t^g_{vfj} \geqslant 0$),

(ii.2) *for each consumer* h, $s^h_{zfj} \leqslant 0$ ($s^h_{vfj} \leqslant 0$), *and*

(ii.3) $p_j \geqslant t^f_{zfj}$ ($-p_j \geqslant t^f_{vfj}$).

(iii) *If the consumption of the jth commodity by the hth producer does not have a general detrimental (beneficial) externality, then*

(iii.1) *for each producer g*, $t^g_{xhj} \leq 0$ ($t^g_{xhj} \geq 0$),

(iii.2) *for each consumer* $i \neq h$, $s^i_{xhj} \leq 0$ ($s^i_{xhj} \geq 0$), *and*

(iii.3) $s^h_{xhj} \leq p_j$ ($s^h_{xhj} \geq p_j$).

Note that (i) and (ii) together imply that if the output of the *j*th commodity at the *f*th producer has no general external effect, then the price received by the *f*th producer for his output of the *j*th commodity is equal to the social price of the *j*th commodity, and the price received by all other agents for the *f*th producer's output of the *j*th commodity is zero. (Similar remarks hold for inputs and for consumers.) Note that the absence of general external effects does not exclude the possibility that the activities of the *f*th producer have external effects on other individual agents. However, in that case detrimental effects must be balanced by beneficial effects in a way that is made precise by definitions (a) and (b) of *general externalities*.

To establish (*i*) let us consider the case of outputs first. The hypothesis implies that there is $z_{fj} > z^*_{fj}$ such that $a^*(z_{fj}) \in T_f$. By (i) and (ii) of Theorem 1, $a^*(z_{fj}) \in T_f$ implies that for $g \neq f$, $t^g \cdot a^*(z_{fj}) \leq t^g \cdot a^*$, and for each h, $-s^h a^*(z_{fj}) \leq -s^h \cdot a^*$. On the other hand, for any z_{fj}, $t^g \cdot a^*(z_{fj}) = t^g \cdot a^* + t^g_{zfj} \cdot (z_{fj} - z^*_{fj})$ for all g, and $s^h \cdot a^*(z_{fj}) = s^h \cdot a^* + s^h_{zfj} \cdot (z_{fj} - z^*_{fj})$ for all h. Hence, for each $g \neq f$, $t^g_{zfj} \cdot (z_{fj} - z^*_{fj}) \leq 0$ and for each h, $s^h_{zfj} \cdot (z_{fj} - z^*_{fj}) \geq 0$. This establishes (i.1) and (i.2), since $z_{fj} - z^*_{fj} > 0$. (i.3) then follows from the fact that $p_j = \sum_{g \in N} t^g_{zfj} - \sum_{h \in M} s^h_{zfj}$. The proof of (i) for inputs is the same, except that (i.3) follows from the fact that $-p_j = \sum_{g \in N} t^g_{vfj} - \sum_{h \in M} s^h_{vfj}$. The proofs of (ii) and (iii) are similar.

The hypotheses of Theorem 1 do not guarantee that there is a price characterization which satisfies (i)–(v) and also has the property that each agent face a nonzero price vector. An example establishing this statement is provided by Murakami and Negishi in [9]. These authors then derive a condition which, when reinterpreted in terms of the model presented here, guarantees that each producer face a nonzero price vector for his own inputs and outputs. This condition is that for each producer, either one of his actually employed inputs is not exerting a beneficial externality or one of his actually produced outputs is not exerting a detrimental externality. By (i.3) and (ii.3) of Theorem 2, this condition (under the above definitions of externality) guarantees that for each commodity j, $p_j \leq t^f_{zfj}$ and $-p_j \geq t^f_{vfj}$. Under Murakami and Negishi's assumption that $p_j > 0$ for each j, we see that their result is obtained in the model under consideration. Similarly (iii.3) implies that if the consumption of the *j*th commodity by the *h*th consumer is not exerting a beneficial external effect, and if $p_j > 0$, then $s^h_{zhj} > 0$.

The proof of Theorem 1 depends heavily upon the convexity assumptions (A.1) and (A.2). As will be discussed in Section 3, these assumptions may be more restrictive in the presence of externalities than otherwise. Also, for certain applications, it may be desirable to have a price characterization similar to that of Theorem 1 for states which may not be globally optimal. We would like to indicate one method which may be helpful both in attempting to relax the convexity assumptions and in obtaining local characterizations. For this we borrow the concept of the tangent cone from Hestenes [6].

Let $\|\cdot\|$ be the Euclidean norm on R^n.

DEFINITION. A sequence $\{x_k\}_{k=1}^{\infty}$ in R^n is said to *converge to* x_0 *in the direction* $h \in R^n$, if

(i) $\|h\| = 1$,

(ii) for each k, $x_k \neq x_0$,

(iii) $\lim_{k\to\infty} \|x_k - x_0\| = 0$,

(iv) $\lim_{k\to\infty} [(x_k - x_0)/\|x_k - x_0\|] = h$.

DEFINITION. Let $S \subseteq R^n$ and $x_0 \in S$. The *tangent cone of* S *at* x_0 is defined as the cone hull of the set of vectors $h \in R^n$ such that there is a sequence in S which converges to x_0 in the direction h. By adding x_0 to each point in the tangent cone of S at x_0, we obtain a set which may serve as a linear approximation to S at x_0.

For a given Pareto-efficient point a^*, let us replace assumption (A.1) with

(A.1)′ For each producer f, the tangent cone of S_f at a^* is convex. Let $\bar{S}_f$ be the tangent cone of S_f at a^* and define $\bar{S}_f = \{a^* + a \mid a \in \bar{S}_f\}$. Then replace (A.4) by

(A.4)′ $\{\bigcap_{h\in M} \mathrm{ri}\, G_h(a^*)\} \cap \{\bigcap_{f\in N} \mathrm{ri}\, \bar{S}_f\} \neq 0$.

In this way we may use Theorem 1 to obtain a shadow price characterization of a^* where the technology sets are replaced by their linear approximations at the Pareto-efficient point. We may consider this a local shadow price characterization. Since (A.1) is satisfied for a large class of nonconvex technology sets S_f, this local characterization may be helpful in dealing with situations in which the full convexity assumption is inappropriate.

3. EXTERNALITIES AND CONVEXITY

In a recent paper [13], Starrett argued that the assumption that there are detrimental externalities in production is inconsistent with other assumptions commonly made concerning production technology; in particular, with certain convexity assumptions. The purpose of this section is to investigate this question in detail.

We shall continue to use the notation and definitions of Section 1 and for the remainder of this section allow f to be a fixed element of N. We no longer make the assumptions of Section 2, although (A.1) is reintroduced as

(B.1) S_f is convex.

Starrett considers a single-product firm and the effects of an externality on its level of output, given a fixed level of inputs (see [13, p. 189].) He implicitly assumes that with these inputs the producer can produce at least zero output regardless of the level of externality. We wish to make this assumption explicit and to generalize it to consider the possibility of multiproduct firms. Hence, we assume

(B.2) there is $(\bar{v}_f, \bar{z}_f) \in R^{2l}$ such that, for all $a \in D$, $(\bar{v}_f, \bar{z}_f) \in S_f(a)$.

For the case of a single-product firm this is slightly weaker than Starrett's assumption, in that the output level $\bar{z}_f$ is not assumed to be 0.

In order to investigate the consistency of (B.1) and (B.2) with the existence of detrimental externalities, a definition of detrimental externality must be adopted. In order to demonstrate that there are alternative reasonable definitions under which we get conflicting answers to the question under consideration, we consider a simple example in which there are no consumers, $l = n = 2$, and S_1 is the set of states $a = (v_{11}, v_{12}, v_{21}, v_{22}; z_{11}, z_{12}, z_{22}) \in D$ which satisfy $z_{11} \leq v_{12}^{1/2}$, and $v_{12} \geq v_{21}$ implies $z_{11} < v_{12}^{1/2}$ (see Fig. 1). S_1 satisfies (B.1) and (B.2). However, since $v'_{21} > v_{21}$ implies that $S_1(a(v'_{21}))$ is a proper subset of $S_1(a(v_{21}))$, there are quite natural definitions of detrimental externalities under which such externalities would be said to exist for S_1. Thus, if such a definition were to be adopted we would conclude that the

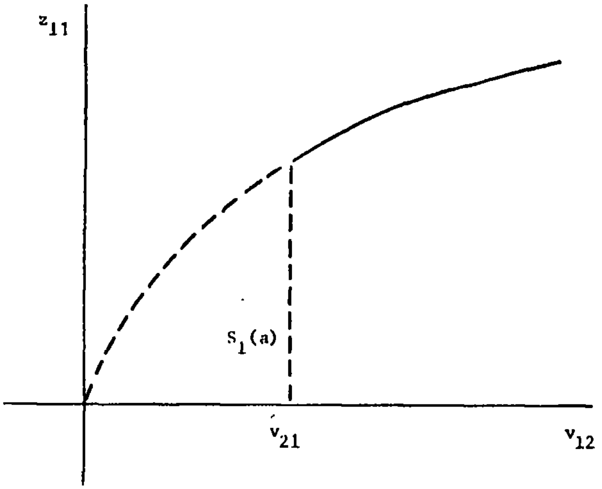

FIG. 1. $S_1(a)$ for a with $v_{21} = \hat{v}_{21}$.

inconsistency conjectured by Starrett does not exist. That there are also reasonable definitions under which the inconsistency could be established will be made clear below.

The detrimental effect existing in the example above is very weak. While an increase in v_{21} does result in a loss of some efficient net output vectors from the possibilities of producer 1, there remain vectors arbitrarily close to those lost. This possibility can be excluded by assuming that

(B.3) for all $a \in D$, $S_f(a)$ is closed.

By including (B.3) it is possible to adopt a very weak definition of externality and show that the existence of detrimental externalities is inconsistent with (B.1), (B.2), and (B.3).

DEFINITION. The output of the jth commodity at the gth producer has a *detrimental external effect* on producer f if, for some $a \in D$ and some $z^2_{gj} > z^1_{gj}$, there is $(\hat{v}_f, \hat{z}_f) \in S_f(a(z^1_{gj}))$ such that $(\hat{v}_f, \hat{z}_f) \notin S_f(a(z^2_{gj}))$. We then consider the assumption that

(B.4) the output of the jth commodity at the gth producer has a detrimental external effect on producer f.

THEOREM 3. *Assumptions* (B.1), (B.2), (B.3), *and* (B.4) *are inconsistent.*

To prove this we suppose that all four assumptions hold for S_f. By (B.4) there are $a \in R^l$, real numbers $z^2_{gj} > z^1_{gj}$, and $(\hat{v}_f, \hat{z}_f) \in R^{2l}$ such that $(\hat{v}_f, \hat{z}_f) \in S_f(a(z'_{gj}))$ and $(\hat{v}_f, \hat{z}_f) \notin S_f(a(z^2_{gj}))$. By (B.2) there is $(\bar{v}_f, \bar{z}_f)$ such that for all $a \in D$, $(\bar{v}_f, \bar{z}_f) \in S_f(a)$. Since $(\hat{v}_f, \hat{z}_f) \notin S_f(a(z^2_{gj}))$, (B.3) implies that there is $\epsilon > 0$ such that $\|(v_f, z_f) - (\hat{v}_f, \hat{z}_f)\| < \epsilon$ implies that $(v_f, z_f) \notin S_f(a(z^2_{gj}))$. Let $\delta = \epsilon \|(\bar{v}_f, \bar{z}_f) - (\hat{v}_f, \hat{z}_f)\|^{-1}$, and note that, since $(\bar{v}_f, \bar{z}_f) \in S_f(a(z^2_{gj}))$, $\delta < 1$.

Choose z^3_{gj} large enough so that $0 < \lambda < \delta$, where $\lambda = (z^2_{gj} - z^1_{gj})(z^3_{gj} - z^1_{gj})^{-1}$. Since $(\hat{v}_f, \hat{z}_f) \in S_f(a(z^1_{gj}))$ and $(\bar{v}_f, \bar{z}_f) \in S_f(a(z^3_{gj}))$, (B.1) implies that $(\dot{v}_f, \dot{z}_f) = \lambda(\bar{v}_f, \bar{z}_f) + (1 - \lambda)(\hat{v}_f, \hat{z}_f) \in S_f(a(\lambda z^3_{gj} + (1 - \lambda) z^1_{gj})) = S_f(a(z^2_{gj}))$, the last equality following from the definition of λ. We derive a contradiction by showing that the distance from $(\dot{v}_f, \dot{z}_f)$ to $(\hat{v}_f, \hat{z}_f)$ is less than ϵ, from which it follows that $(\dot{v}_f, \dot{z}_f) \notin S_f(a(z^2_{gj}))$. Indeed, $\|(\dot{v}_f, \dot{z}_f) - (\hat{v}_f, \hat{z}_f)\| = \lambda \|(\bar{v}_f, \bar{z}_f) - (\hat{v}_f, \hat{z}_f)\| < \delta \|(\bar{v}_f, \bar{z}_f) - (\hat{v}_f, \hat{z}_f)\| = \epsilon$.

Although we have considered explicitly only the case where the output of one producer has a detrimental effect, it is easy to see that the same argument will establish the inconsistency in the cases where the detrimental effect results from the input of a commodity by a producer, or the consumption of a commodity by a consumer.

An alternative approach to the general question being considered in this section is to drop the explicit convexity assumption (B.1) and ask if (B.2) is consistent with the existence of an equilibrium on artificial markets for which there are positive "externality" prices. The following theorem provides a quite definite answer.

THEOREM 4. *If S_f satisfies assumption* (B.2) *and if t^f is a price system for which there is $g \neq f$ such that t^f_{zg} has a positive component, then the profit function $t^f \cdot a$ is unbounded on S_f.*

To prove this, suppose that t^f_{zg} has a positive component and let m be any positive number. Let $(\bar{v}, \bar{z})$ be as in (B.2). There is $\hat{z}_g \in R_+^l$ such that $t^f_{zg} \cdot \hat{z}_g = m - t^f_{vf} \cdot \bar{v}_f - t^f_{zf} \cdot \bar{z}_f$. By (B.2), $\hat{a} = (0, \ldots, 0; 0, \ldots, \bar{v}_f, \ldots, 0; 0, \ldots, \hat{z}_g, \ldots, \bar{z}_f, \ldots, 0) \in S_f$. This establishes the desired result, since $t_f \cdot \hat{a} = t^f_{vf} \cdot \bar{v}_f + t^f_{zf} \cdot \bar{z}_f + t^f_{zg} \cdot \hat{z}_g = m$.

We can in fact utilize Theorem 4 to establish a result very similar to that of Theorem 3.

COROLLARY. *Suppose that S_f is closed and satisfies* (B.2). *Let a^* be an efficient point of S_f and suppose there is a local shadow price, t^f, associated with a^*. If for some $g \neq f$, $t^f_{zg} > 0$, then S_f is not convex.*

This follows from the fact that if S_f is convex then a^* maximizes $t^f \cdot a$ on S_f, which is impossible by Theorem 4.

If we consider a model in which production allocations are represented by vectors of net outputs, the difficulties displayed by Theorems 3 and 4 exist not only for detrimental externalities, but for beneficial externalities as well. To verify this, simply note that if there are beneficial externalities (defined in a manner symmetrical to the definitions of detrimental externalities), we would, in the proof of Theorem 3, have $z^1_{gj} > z^2_{gj}$, $(v_f, z_f) \in S_f(a(z^2_{gj}))$, and $(v_f, z_f) \notin S_f(a(z^1_{gj}))$. Then z^3_{gj} could be chosen negative and large in absolute value, and the same argument would lead to a contradiction. Similarly, in Theorem 4, if $t_g{}^f$ has a negative component, the vector z_g can be chosen negative and arbitrarily large in absolute value, again showing the unboundedness of the profit function. For the case of detrimental externalities Starrett argues intuitively that if a producer faces a positive externality price for the net output of a certain commodity by another producer, he will demand arbitrarily large positive quantities of that net output, since such output can result in only "finite damage" for the producer in question. But if the same producer faces a negative externality price, he will, under the same reasoning, demand arbitrarily large (in absolute value) negative quantities.

For the case of beneficial externalities, the inconsistencies under consideration can always be eliminated by measuring absolute levels of inputs and outputs instead of merely measuring net outputs, for then a given producer cannot demand negative quantities of inputs or outputs from other producers. Thus, while some care must be taken in the case of beneficial externalities, Starrett seems to be correct in asserting that the problem is fundamental to the consideration of detrimental externalities and not externalities in general.

Without (B.2), detrimental externalities can be perfectly compatible with full convexity of the technology sets. (That is, (B.1), (B.3), and (B.4) are not inconsistent.) Consider, for example, a simple economy in which there are no consumers $(D = R_+^{2nl})$, $l = 3$, $n = 2$, $S_1 = [(v_1, v_2, z_1, z_2) \in D \mid z_{11} \leq v_{13} - z_{22}]$, and $S_2 = [(v_1, v_2, z_1, z_2) \in D \mid z_{22} \leq v_{23}]$ (see Fig. 2). Such a representation of technology might be used to analyze a situation in which the second producer's output of commodity two involves the dumping of some waste material which the first producer must eliminate in order to carry on his own production activities. Although detrimental externalities play an essential role in this technology, both S_1 and S_2 are convex and all efficient points can be supported by prices on artificial markets. (It can easily be checked that all efficient aggregate output allocations can actually be supported by the same price system, consisting of $p = (1, 2, 1)$, $t^1 = (-1, -2, -1, 0, 0, 0, 1, 2, 1, 0, 1, 0)$, and $t^2 = (0, 0, 0, 1, 1, 1, 0, 0, 0, -1, -2, -1)$.)

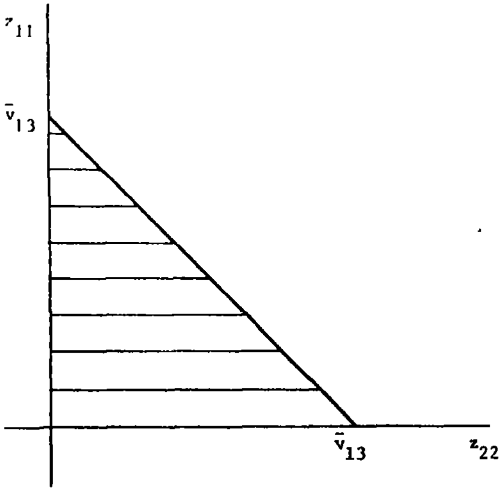

FIGURE 2

Other examples can be constructed by considering a firm whose production of a single output, z_{11}, is governed by a concave production function, $f(v_1)$, but whose maximum possible use of an input, v_{12}, is reduced by the output of a pollution product, z_{23}, of a second firm. That is, let $A > 0$ and $S_1 = \{v_1, v_2, z_1, z_2) \in D \mid z_{11} \leq f(v_{12})$ and $v_{12} \leq A - z_{23}\}$ (see Fig. 3). If $z_{23} > A$ then the set of input-output combinations (v_1, z_1) which are possible for producer 1 is empty. S_1 is convex. However, if the entire z_{22} axis is considered technologically feasible, i.e., S_1 is defined as $\{(v_1, v_2, z_1, z_2) \in D \mid z_{11} \leq f(v_{12})$ and $v_{12} \leq \max(A - z_{23}, 0\}$, then S_1 is clearly not convex.

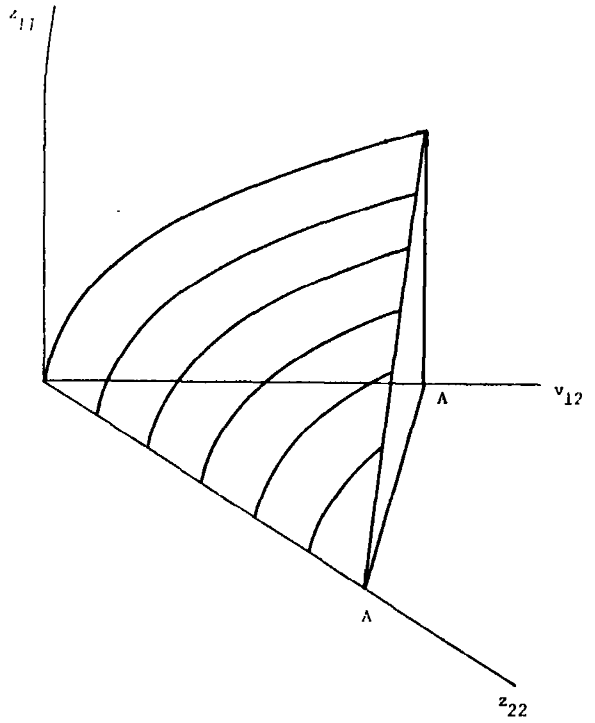

FIGURE 3

Note that these examples also show that convexity and the existence of detrimental externalities are not necessarily inconsistent with assumptions which could be used, alternatively to (B.2), to represent the possibility of shutting down or remaining inactive. For example, we could represent inactivity as the choice of no point in the technology set, or the choice of the empty set. Profits resulting from this choice would be defined as zero. (If a producer remains inactive he does not enter the market at all and has no chance to collect externality payments.) We could then assume that when faced with a given price system the producer will remain inactive unless he can choose a point in his technology set at which he would earn nonnegative profits. In the above example such an assumption would not prevent any efficient aggregate output allocation from being chosen under the given price system.

Thus there are inconsistencies only when (B.2) is an appropriate assumption. Starrett considered an economy in which producers are motivated by a desire for profits and can allow waste materials to accumulate indefinitely without penalty. In such cases and when the model is being used for the purpose of prediction, (B.2) may be a useful behavioral assumption. However, if the model is being used to analyze the possibilities for partially decentralized planning procedures, we are more concerned with a characterization of the technological possibilities and (B.2) seems to be no longer appropriate. The potential of the firm to initiate production activities does not exist independently of the activity of other agents, as (B.2) would suggest. Even if a producer abandons his plant the economy still possesses a production potential, and the nature of that potential may vary with the activity of other agents. On the other hand, if (B.2) is intended to represent the possibility that a producer can pretend to be able to maintain his production potential even when he cannot, then the problem becomes a matter of the producer's incentive to incorrectly reveal the nature of his technology. This is always a problem when dealing with externalities and corrective schemes and is not fundamentally related to convexity or other properties of the true technology.

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Stability of Social Choices in Infinitely Large Societies

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1. INTRODUCTION

Arrow's conditions for social welfare functions have been shown by Fishburn [4] and by Kirman and Sondermann [6] to be consistent with the hypothesis that the set of individuals is infinite. Closely related to Arrow's impossibility theorem [1] are the recent results obtained by Gibbard [5] and Satterthwaite [9], according to which every social choice function that is immune to manipulation (misrepresentation of preferences) by a single individual is either imposed or dictatorial whenever the number of individuals is finite. It is natural to enquire whether the apparent duality in the finite case between Arrow's negative results and those of Gibbard and Satterthwaite carries over to the case of an infinite set of individuals. Not surprisingly the answer is affirmative. When the set of individuals is infinite it is not difficult to exhibit an example of a nondictatorial social choice function that is immune to manipulation by individuals (such an example is provided in Section 2). This fact is in agreement with the commonly held notion that in large societies a state of stability can be approached which is not likely to be disturbed by the acts of any individual.

A greater threat to stability under these circumstances would stem from the emergence of concerted actions among sizable coalitions. It is therefore of interest to investigate situations in which stability remains intact when joint strategies of groups are also taken into account. In the present context, where we deal with voting, we are led to investigate the effects of misrepresentation of preferences by entire coalitions. We ask whether a coalitionally nonmanipulable social choice function is possible when the number of voters is infinite. The existence of a nonimposed, nondictatorial social choice function F with this property is the subject of the main theorem in Section 2.

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The social structure engendered by the function F produced below, though nondictatorial, possesses (as is explained in Section 3) some features which are characteristic of oligarchical societies. Suppose, for example, that the set of individuals (voters) V corresponds to the points in the unit interval $[0, 1]$. Then under the social choice function F there are arbitrarily small subintervals of individuals whose block-votes determine the social choice regardless of the votes of outsiders. That is, for any $\epsilon > 0$ there is a subinterval (a, b) of length ϵ whose members decide the social choice whenever they vote identically. The "ruling" subintervals all intersect one another. At the same time, no individual ever has any noticeable influence in determining the final outcome. In this situation, the society V can be seen as being governed by a vanishingly small elite. On the other hand, an almost paradoxical equality prevails between any two voters in view of their insignificance in influencing the social choice. This feature also motivates the discussion of participatory voting in the next section. In particular, the difference between the notions of incentive compatibility and cheatproofness in the social choice context is examined in Section 4.

In the last section we consider the case where the set of alternatives is infinite. We then inquire whether a coalitionally nonmanipulable, nonimposed, and nondictatorial social choice function exists which is also Pareto. It appears that in order for a social choice function with these properties to be possible, the order of infinity (cardinality) of V would have to be extremely large.

2. EXISTENCE OF COALITIONALLY-CHEATPROOF SOCIAL CHOICE FUNCTIONS

Let $\mathcal{Cl}$ denote a finite set of alternatives and let Σ denote the class of strong orderings (complete, asymmetric, and transitive binary relations) on $\mathcal{Cl}$. Let V denote an infinite set of individuals. A *Social Choice Function* (SCF) F is a function $F: \Sigma^V \to \mathcal{Cl}$. An element in the domain of F is called a *preference profile* and is denoted by $\{P_i\}_{i \in V}$ where $P_i \in \Sigma$ and $i \in V$. An SCF is *individually-manipulable* at $\{P_i\}_{i \in V} \in \Sigma^V$ if there exists a $\{P_i'\}_{i \in V} \in \Sigma^V$ such that for some i_0, $F(\{P_i'\}_{i \in V}) \, P_{i_0} F(\{P_i\}_{i \in V})$ where $P_i' = P_i$ for all $i \neq i_0$. An SCF F is *individually cheatproof* if there is no preference profile at which it is individually manipulable; F is *dictatorial* if there exists a j in V such that for all profiles $\{P_i\}_{i \in V}$ and for all $x \neq F(\{P_i\}_{i \in V})$ in the range of F, $F(\{P_i\}_{i \in V}) \, P_j x$; F is *imposed* if its range is a singleton. Finally, F will be said to satisfy the *unanimity* criterion if whenever for all i, $x P_i a$ for all $a \neq x$ in $\mathcal{Cl}$, then $F(\{P_i\}_{i \in V}) = x$. As shown by Gibbard [5] and Satterthwaite [9] (a simple proof of which is provided by Schmeidler and

Sonnenschein in [10]), when the number of individuals is finite there exists no individually cheatproof and nondictatorial SCF if the range of F contains at least three alternatives. As shown by the following example, this impossibility result does not carry over to the case of an infinite set of individuals.

Let $\mathcal{A} = \{a_1, a_2, \ldots, a_m\}$ and let V be infinite. Set $F(\{P_i\}_{i \in V}) = a_j$ $(j = 1, 2, \ldots, m)$ where j is the first index for which $\bar{D}(a_j, \{P_i\}_{i \in V}) \overset{\text{def}}{=} \{i \in V \mid a_j$ is maximal with respect to $P_i\}$ is infinite. This F is clearly not dictatorial; it is not imposed since it satisfies the unanimity criterion. Since no single individual can by himself alter the finiteness or infiniteness of any $\bar{D}(a_j, \{P_i\}_{i \in V})$ $(j = 1, 2, \ldots, m)$ this F is clearly individually cheatproof. This completes our example.

We turn now to the definition of coalitional cheatproofness. An SCF $F: \Sigma^V \to \mathcal{A}$ is said to be *coalitionally-manipulable* if for some profile $P = \{P_i\}_{i \in V}$ and some nonempty subset $E \subset V$ and some profile $P' = \{P_i'\}_{i \in V}$ where $P_i' = P_i$ for all $i \notin E$, $F(P') P_i F(P)$ for all $i \in E$. If F is not coalitionally-manipulable it is said to be *coalitionally-cheatproof*.

The main object of the present paper is to establish that when the set of individuals is infinite there exists a nonimposed and nondictatorial SCF that is coalitionally cheatproof.

Before proceeding to the main theorem, we cite some definitions and results which will be used in the proof below. For proofs of these results, see Bourbaki [2].

DEFINITION 1. Let V denote an infinite set. A nonempty set $\mathcal{F}$ of subsets of V is a *filter* on V if each of the following conditions are met:

(a) Any subset of V which includes an element of $\mathcal{F}$ belongs to $\mathcal{F}$.

(b) Any finite intersection of elements of $\mathcal{F}$ belongs to $\mathcal{F}$.

(c) The empty set does not belong to $\mathcal{F}$.

DEFINITION 2. A set $\mathcal{U}$ of subsets of V is an ultrafilter on V if:

(a) $\mathcal{U}$ is a filter, and

(b) for every subset S of V, either $S \in \mathcal{U}$ or $S^{\text{comp}} \in \mathcal{U}$ (where $S^{\text{comp}} = V \backslash S$).

DEFINITION 3. A filter $\mathcal{F}$ over V is a *principal filter* if there exists an $i \in V$ such that $\{i\} \in \mathcal{F}$. If no such i exists, $\mathcal{F}$ is called a *non-principal filter*.

EXAMPLE. Let $\mathcal{F}$ be the set of subsets S of V such that S^{comp} is finite: $\mathcal{F}$ is then a non-principal filter.

PROPOSITION 1. (See Bourbaki [2] for proof). *Given any filter $\mathcal{F}$ over V, where V is an infinite set, there exists an ultrafilter $\mathcal{U}$ such that for any $S \in \mathcal{F}$, S is also in $\mathcal{U}$.*

PROPOSITION 2. *For any finite partition $V_1, V_2, \ldots, V_n$ of an infinite set V, $(\bigcup_{i=1}^{m} V_i = V$, $V_i \cap V_j = \varnothing$ for all $i \neq j)$, if $\mathcal{U}$ is an ultrafilter on V then there exists an i_0 $(i_0 = 1, \ldots, n)$ such that $V_{i_0} \in \mathcal{U}$.*

We now have the necessary equipment to prove the following:

THEOREM. *If V is infinite and $\mathcal{A}$ is finite, then there exists a coalitionally-cheatproof* SCF $F\colon \Sigma^V \to \mathcal{A}$ *which is neither imposed nor dictatorial.*

Proof. Let $\mathcal{U}$ be a non-principal ultrafilter on V. For all $a \in \mathcal{A}$ and all $P = \{P_i\}_{i \in V} \in \Sigma^V$, let $D(a, P) \overset{\text{def}}{=} \{v \in V \mid a$ is maximal with respect to $P_v\}$ and consider the sets $D(a_1, P), D(a_2, P), \ldots, D(a_m, P)$, where m is the number of elements in $\mathcal{A}$. Since $\bigcup_{i=1}^{m} D(a_i, P) = V$ and $D(a_i, P) \cap D(a_j, P) = \varnothing$ for any $i \neq j$ $(i, j = 1, 2, \ldots, m)$ there exists an i_0 such that $D(a_{i_0}, P)$ belongs to $\mathcal{U}$. Define the SCF F by $F(P) = a_{i_0}$. Since $\mathcal{U}$ is nonprincipal, F is clearly nondictatorial; it is not imposed since it satisfies the unanimity condition.

We prove now that F is coalitionally-cheatproof. Suppose to the contrary that F is coalitionally-manipulable. Then for some $P \in \Sigma^V$ there exists an $E \subset V$ and some $P' \in \Sigma^V$ where $P_i' = P_i$ for all $i \notin E$, such that $F(P')\, P_i F(P)$ for all $i \in E$. Let $a = F(P)$ and $a' = F(P')$. Since $E \neq \varnothing$, $F(P) \neq F(P')$ and thus $a \neq a'$. Let $C \overset{\text{def}}{=} D(a, P)$. Then $C \in \mathcal{U}$ and there are two possibilities: (a) $E \cap C \neq \varnothing$; (b) $E \cap C = \varnothing$. Consider possibility (a). If $j \in E \cap C$ then since $j \in C$, $F(P)\, P_j F(P')$ implying $j \notin E$, a contradiction. Consider the possibility (b). By definition $F(P') = a'$ where a' satisfies $C' = D(a', P') \in \mathcal{U}$. But $C' \cap C \neq \varnothing$, since both $C' \in \mathcal{U}$ and $C \in \mathcal{U}$. So let $j \in C' \cap C$. Then $a' P_j' b$ for all $b \neq a'$ $(b \in \mathcal{A})$, since $j \in C'$. Now since $E \cap C = \varnothing$, $j \notin E$ (since $j \in C$) and thus $P_j' = P_j$ by the definition of E. Therefore $a' P_j b$ for all $b \neq a'$ $(b \in \mathcal{A})$ and thus in particular $a' P_j a$, a contradiction to $j \in C$. This completes the proof. We remark that if the strong ordering assumption governing the preference orderings is relaxed and we permit the profiles to include weak orderings as well, then the theorem remains valid.

3. THE SIMULATION OF OLIGARCHICAL SOCIETIES

The SCF $F\colon \Sigma^V \to \mathcal{A}$ defined in the preceding theorem, though nondictatorial generates a social order over V which is strongly reminiscent of an institutional hierarchy. If V corresponds to the set of reals in [0 1] and F is an SCF defined as above by means of an ultrafilter over V then the following is true: For any $\epsilon > 0$ there exists a subinterval $[a_1, a_2]$ such that $(a_2 - a_1) < \epsilon$ and such that if the individuals in $[a_1, a_2]$ vote in block then under the SCF F these voters get their way regardless of the actions of those outside the interval. The "decisive intervals" all intersect one another. At the same time, no individual has any influence over the final social outcome. For any voter $x \in [0, 1]$, if x's vote is counter to the wishes of all (other) members of some ruling interval, then x is ignored.

Is there an invisible dictator? Kirman and Sondermann [6], in dealing with the social welfare function of Fishburn's work [4], maintained that even in the infinite case an Arrovian social welfare function must be dictatorial in some sense. Although the arguments of Kirman and Sondermann can be carried over to the present situation as well, a more appropriate interpretation in this context would appear to be that a vanishingly small elite rather than an invisible dictator exists.

In other situations, coalitionally-cheatproof SCF's can produce a somewhat more democratic social structure. Consider the following example: Let V_1 be the set of reals in $(-\infty, \infty)$. Then an ultrafilter $\mathcal{U}_1$ over V_1 exists with the following property: Let $F_1 \colon \Sigma^V \to \mathcal{A}$ be defined by means of $\mathcal{U}_1$ as in the theorem above; then for any finite interval (c_1, c_2) if all voters outside the interval unite behind a single alternative $a \in \mathcal{A}$, then the SCF F_1 chooses that alternative, regardless of the voters inside (c_1, c_2).

4. THE PROBLEM OF PARTICIPATION

A disturbing feature of the social choice function dealt with here is the salient lack of individual motivation to vote, since no individual is ever able to influence the social outcome. The recurrent question of "why bother to vote" stands out quite sharply in the present context. Actually the analysis here may help to shed some light on the difference between incentive compatibility and cheatproofness in the social choice context.

In the standard Arrow–Debreu model of an economy with pure private goods only, the requirement of cheatproofness is essentially equivalent to that of incentive compatibility. For, in such an economy even when any single agent has no influence on prices (as is the case for instance when there is a continuum of atomless agents) he clearly does have an incentive to "vote" (e.g., to state his excess-demand under the competitive mechanism) since this is the only way for him to get his most preferred bundle. However, if public goods are present in such an atomless economy, since no single agent can have any influence on the amounts of public goods provided, the "free rider" problem (i.e., lack of incentive compatibility) comes into play. Note that cheatproofness is not an issue here since the lack of motivation to "vote" implies in particular a lack of motivation to cheat.

In the social choice context, the closest analogy to an economic model would be to a pure public goods economy in which no single commodity is individually appropriable. This follows from the observation that the set of alternatives is common to all and that unrestricted domain of preferences is assumed. In such a case, when the set of voters is infinite, cheatproofness again results from the fundamental lack of individual incentives to participate in the voting game.

The problem of participatory motivation, though particularly acute in the infinite case is present in the finite case as well. As the existing cheatproofness literature has essentially ignored the problem of incentive compatibility in this participatory sense, this problem arises there as well.

5. Social Choice Functions with Infinitive Alternatives

Having shown the existence of a coalitionally-cheatproof nondictatorial SCF $F\colon \Sigma^V \to \mathcal{A}$ when the set of alternative $\mathcal{A}$ is finite, we inquire whether the theorem carries over for the case where $\mathcal{A}$ is infinite. It is easily seen that it does. For if $\mathcal{A}$ is infinite, let $\mathcal{A}' \subset \mathcal{A}$ contain only three alternatives and let Σ_0 be the set of preferences on $\mathcal{A}'$. As before, let Σ be the set of preferences on $\mathcal{A}$. For each $p \in \Sigma$, let $\mathring{p} \in \Sigma_0$ be the preference on $\mathcal{A}'$ obtained by restricting p to $\mathcal{A}'$. For each $P = \{P_i\}_{i\in V} \in \Sigma^V$, let $\mathring{P} = \{\mathring{P}_i\}_{i\in V} \in \Sigma_0{}^V$. In accordance with the foregoing theorem there exists a nonimposed, nondictatorial, coalitionally-cheatproof SCF F_0: $\Sigma_0{}^V \to \mathcal{A}'$. For each $P \in \Sigma^V$, let $F(P) = F_0(\mathring{P})$. It is then easily seen that $F\colon \Sigma^V \to \mathcal{A}$ is nonimposed, nondictatorial, and coalitionally-cheatproof. However, F is not Pareto. (We say that an SCF $F\colon \Sigma^V \to \mathcal{A}$ is Pareto if there does not exist $a, b \in \mathcal{A}$ and a profile $P = \{P_i\}_{i\in V}$ such that $F(\{P_i\}_{i\in V}) = a$ and for all $i \in V$, bP_ia.) Clearly, unless restrictions are made on the profiles, no Pareto SCF exists. (Consider a profile $P = \{P_i\}_{i\in V}$ where $P_i = \bar{p}$ ($\bar{p}$ is some fixed preference order) for all $i \in V$, and $\bar{p}$ contains no maximal element.) We are therefore interested in whether a Pareto SCF, F, exists over the set of profiles $\{P = \{P_i\}_{i\in V} \mid \text{each } P_i \text{ contains a maximal element}\}$. More precisely, suppose $\mathcal{A}$ is countably infinite and let Σ_1 be the set of complete preference orders over $\mathcal{A}$ that have maximal elements. Let V be an infinite set of voters. We inquire as to whether there is a coalitionally-cheatproof, nondictatorial Pareto SCF $F\colon \Sigma_1{}^V \to \mathcal{A}$.

Under the assumption that $\mathcal{A}$ is countably infinite it may then be shown

(Proposition A1, see Appendix) that if V is countably infinite, no such SCF exists. (This and all that follows remains true if $\mathcal{A}$ is of the same cardinality as the continuum.) It is then natural to ask whether a coalitionally-cheatproof. nondictatorial, Pareto, SCF $F: \Sigma_1^V \to \mathcal{A}$ exists if V is larger, i.e., if V is uncountable. Using considerations similar to those of Ulam [12] one may in fact show that no such SCF is possible whenever V is of cardinality $\aleph_1, \aleph_2, \aleph_3, \ldots$ or even $\aleph_{\aleph_0}$. This will be seen as a consequence of the discussion which soon follows. The question then remains, does there exist any large enough V for which a coalitionally-cheatproof, nondictatorial, Pareto SCF $F: \Sigma_1^V \to \mathcal{A}$ exists. The answer depends upon whether a set of measurable cardinality exists.[1] To see this connection, suppose that V is an infinite set of voters for which a coalitionally-cheatproof, nondictatorial, Pareto SCF $F: \Sigma_1^V \to \mathcal{A}$ exists, where $\mathcal{A}$ is countably infinite. Given an SCF $F: \Sigma_1^V \to \mathcal{A}$ with these properties, let $\mathcal{U}_1$ be the following set of subsets of V:

(1) $V \in \mathcal{U}_1$.

(2) $W \subset V$ is in $\mathcal{U}_1$ iff there exists a pair of preferences $p', p'' \in \Sigma$ with maximal elements $a', a'', a' \neq a''$ such that for the profile $P = \{P_i\}_{i \in V}$ where $P_i = p'$ for $i \in W$, $P_j = p''$ for $j \in W^{\text{comp}}$, $F(P) = a'$.

It may be shown that $\mathcal{U}_1$ is a non-principal ultrafilter (Proposition A2, see Appendix). Moreover, if $A_1, A_2, \ldots$ is any countable sequence of sets belonging to the ultrafilter, then $\bigcap_{i=1}^{\infty} A_i$ is also in $\mathcal{U}_1$ (Proposition A3). Thus V must be of measurable cardinality. Conversely, if V is of measurable cardinality then it may be shown that a Pareto, nondictatorial, coalitionally-cheatproof SCF $F: \Sigma_1^V \to \mathcal{A}$ exists (Proposition A4). It is not known whether sets of measurable cardinality exist. It has been shown [12], however, that sets of measurable cardinality, if they exist, are very large. Even if they do exist, their existence cannot be proven within Zermelo–Fraenkel set theory.

One may thus be inclined to believe that for all practical purposes, there is no set of voters V large enough for a coalitionally-cheatproof, nondictatorial, Pareto SCF $F: \Sigma_1^V \to \mathcal{A}$ to exist. This assessment would doubtless be acceptable from the point of view of the social scientist. Mathematically, however, the question of whether or not a set V exists having this attribute (and which would thus be of measurable cardinality) has been of considerable interest in certain analytical problems. In [7], an affirmative solution to the Borel determinateness problem in game theory is produced under the assumption that measurable cardinals exist. Until recently, no alternative proof was known that did not make similar suppositions.

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[1] A set C is said to be of measurable cardinality if there exists a non-principal ultrafilter $\mathcal{U}$ over C such that for every countable sequence $A_1, A_2, \ldots$ of subsets of C, where $A_i \in \mathcal{U}$ for all i, the set $\bigcap_{i=1}^{\infty} A_i$ also belongs to $\mathcal{U}$. (This definition of measurable cardinality conforms to the one appearing in [11].)

APPENDIX

PROPOSITION A1. *Let* $\mathcal{A} = \{a_1, a_2, \ldots\}$ *be a countably infinite set of alternatives, and* $V = \{1, 2, \ldots\}$ *be a countably infinite set of voters. Let* Σ *be the set of strong preference orders on* $\mathcal{A}$. *Let* $\Sigma_1 \subset \Sigma$ *be the set of preference orders on* $\mathcal{A}$ *that have maximal elements. If an* SCF $F\colon \Sigma_1{}^V \to \mathcal{A}$ *is Pareto and coalitionally-cheatproof then it is dictatorial.*

Proof. By method of contradiction, suppose that F is Pareto, coalitionally-cheatproof, and nondictatorial. We consider the following lemma.

LEMMA. *Let* $\{p = p_1, p_2, \ldots, p_n, p_{n+1}, \ldots\}$ *be any profile in* $\Sigma_1{}^V$ *such that for some* n, $p_{n+1} = p_{n+2} = \cdots$, *and let* $F\colon \Sigma_1{}^V \to \mathcal{A}$ *be nondictatorial, coalitionally-cheatproof, and Pareto. Then* $F(p) = a$, *where* $a \in \mathcal{A}$ *is maximal with respect to* p_{n+1}.

Proof of Lemma. Let Σ^{n+1} be the set of profiles $(p_1, \ldots, p_{n+1})$ where for each $1 \leqslant i \leqslant n+1$, p_i is a strong preference order over $\mathcal{A}$ with a maximal element. Let $F_{n+1}\colon \Sigma^{n+1} \to \mathcal{A}$ be defined by

$$F_{n+1}(p_1, \ldots, p_n, p_{n+1}) = F(p_1, p_2, \ldots, p_n, p_{n+1}, p_{n+1}, p_{n+1}, \ldots).$$

Then F_{n+1} is a nonimposed coalitionally-cheatproof SCF defined over profiles of $V_{n+1} =^{\text{def}} \{1, 2, \ldots, n+1\}$. The Gibbard [5] and Satterthwaite [9] theorem (which assumes that V_{n+1} is finite but imposes no upper bound on the size of $\mathcal{A}$) then asserts that F_{n+1} must be dictatorial. If $1 \leqslant i \leqslant n$ is a dictator for F_{n+1}, then this same i is a dictator for F, a contradiction. Hence $n+1$ must be a dictator for F_{n+1}. Thus F depends only on p_{n+1}. Hence $F(p) = F(p_1, p_2, \ldots, p_n, p_{n+1}, p_{n+1}, \ldots) = F(p') = F(p_1', p_2', \ldots, p_n', p_{n+1}, p_{n+2}, \ldots)$ for any $p_1', \ldots, p_n'$. In particular, $F(p) = F(p_{n+1}, p_{n+1}, \ldots, p_{n+1}, p_{n+1}, \ldots) = a$ by the Pareto condition.

To complete the proof of Proposition A1, consider the profile $p = p_1, p_2, p_3, \ldots$ where

$$p_1 = a_1 >_{p_1} a_2 >_{p_1} a_3 >_{p_1} a_4 >_{p_1} a_5 >_{p_1} a_6 >_{p_1} a_7 >_{p_1} \cdots$$

$$p_2 = a_2 >_{p_2} a_1 >_{p_2} a_3 >_{p_2} a_4 >_{p_2} a_5 >_{p_2} a_6 >_{p_2} a_7 >_{p_2} \cdots$$

$$p_3 = a_3 \; p_3 \; a_2 \; p_3 \; a_1 \; p_3 \; a_4 \; p_3 \; a_5 \; p_3 \; a_6 \; p_3 \; a_7 \; p_3 \cdots$$
$$p_4 = a_4 \; p_4 \; a_3 \; p_4 \; a_2 \; p_4 \; a_1 \; p_4 \; a_5 \; p_4 \; a_6 \; p_4 \; a_7 \; p_4 \cdots$$
$$\vdots$$

Then $F(p) = a_n$ for some n. Let $p_1' = p_1$, $p_2' = p_2, \ldots, p_n' = p_n$, and let

$$p'_{n+1} = a_{n+1} \; p'_{n+1} \; a_n \; p'_{n+1} \cdots p'_{n+1} \; a_1 \; p'_{n+1} \; a_{n+2} \; p'_{n+1} \; a_{n+3} \; p'_{n+1} \cdots$$

and let $p'_{n+j} = p'_{n+1}$ for all positive j. Consider $F(p') = F(p_1', p_2', \ldots)$. In accordance with the lemma $F(p') = a_{n+1}$. Thus for all i in the coalition $E_{n+2} = \{n+2, n+3, \ldots\}$, $F(p') \, p_i \, F(p)$, which contradicts coalitional-cheatproofness. The contradiction shows that F must be dictatorial.

PROPOSITION A2. *Let $\mathcal{A}$ be a countably infinite set of alternatives and V an infinite set of voters. Let Σ_1 be the set of preference orders over $\mathcal{A}$ that have maximal elements. Suppose there exists a nondictatorial, Pareto, coalitionally-cheatproof* SCF $F\colon \Sigma_1{}^V \to \mathcal{A}$. *Given such an F, let $\mathcal{U}$ be the following family of subsets $W \subset V$:*

(1) $W = V \in \mathcal{U}$;

(2) $\varnothing \neq W \subset V$ *is in* $\mathcal{U}$ *iff there exists a pair of preferences* $P', P'' \in \Sigma_1$ *with maximal elements* a', a'', $a' \neq a''$ *such that for the profile* $p = \{p_i\}_{i \in V}$ *with* $p_i = P'$ *for* $i \in W$, $p_j = P''$ *for* $j \notin W$, $F(p = \{p_i\}_{i \in V}) = a'$.

Then $\mathcal{U}$ is a non-principal ultrafilter.

Proof. We must prove: (a) For each $W_1, W_2 \in \mathcal{U}$, $W_1 \cap W_2 \neq \varnothing$ and $W_1 \cap W_2 \in \mathcal{U}$, and (b) for every $W \subset V$, either $W \in \mathcal{U}$ or $W^{\text{comp}} \in \mathcal{U}$.

(a) Suppose $W_1, W_2 \in \mathcal{U}$ and $W_1 \cap W_2 = \varnothing$. Since $W_1 \in \mathcal{U}$, this implies the existence of $P', P'' \in \Sigma_1$ with maximal elements a', a'', $a' \neq a''$ such that for $p = \{p_i\}_{i \in W} \in \Sigma_1{}^V$ where $p_i = P'$ for $i \in W_1$, $p_j = P''$ for $i \notin W_1$, $F(p) = a'$. However, from Gibbard and Satterthwaite's result about finite sets of voters it follows directly that under F, if the coalitions W_1, W_1^{comp} each vote in a single block, one of them must dictate. The fact that W_1 gets its way for P' implies that W_1 dicates. We further assert that if members of W_1 vote as a block, they dicate even if members of W_2 do not vote as a block. Otherwise coalitional-cheatproofness would be violated. (To see this consider a P''' in Σ_1 which has a' as *minimal* element, where a' is at the same time the maximal element of P'. Let $\hat{p} = \{\hat{p}_i\}_{i \in V}$ be defined by $\hat{p}_i = P'$ for $i \in W_1$, $\hat{p}_j = P'''$ for $i \notin W_1$. If for some $\bar{p} = \{\bar{p}_i\}_{i \in V}$, where $\bar{p}_i = p_i$ for $i \in W_1$, $F(\bar{p}) \neq a'$, then F is manipulable at $\hat{p}$ for some subset $E' \subset W_1^{\text{comp}}$.) Similarly W_2 must get its way whenever its members vote in block. Since $W_1 \cap W_2 = \varnothing$, W_1 and W_2 can vote different ways. Let W_1 and W_2 each vote in block so that the preference of W_1 has a different maximum than that of W_2. Then W_1 and W_2 must each get their way, a contradiction.

Suppose $W_1 \cap W_2$ were not in $\mathcal{U}$. Consider the coalitions $W_1 - W_2$, $W_1 \cap W_2$, $W_2 - W_1$, each of whose members vote in block. The Gibbard–Satterthwaite theorem then implies that one of these three coalitions dictates under F. $W_1 \cap W_2$ does not dicatate; otherwise it would be in $\mathcal{U}$. If $W_1 - W_2$ dictates then W_2 does not dictate, which violates the assumption that $W_2 \in \mathcal{U}$. Similarly if $W_2 - W_1$ dictates we also get a contradiction. Thus $W_1 \cap W_2 \in \mathcal{U}$.

(b) Let W be any nonempty subset of V. If $W^{\text{comp}} = \varnothing$ then $W = V \in \mathcal{U}$ by definition of $\mathcal{U}$. Suppose $W^{\text{comp}} \neq \varnothing$. Let P', P'' be any two complete preference orders in Σ_1 having maximal elements a', a'', $a' \neq a''$. Let $p = \{p_i\}_{i \in V} \in \Sigma_1{}^V$ be the profile such that $p_i = P'$ for $i \in W$, $p_i = P''$ for $i \in W^{\text{comp}}$. Applying the Gibbard–Satterthwaite result to the present situation, it follows that either W or W^{comp}, when voting in block, must dictate under F. If W dictates then $W \in \mathcal{U}$, if W^{comp} dictates the $W^{\text{comp}} \in \mathcal{U}$.

Proposition A3. *Let V, F, and $\mathcal{U}$ be defined as in Proposition* A2. *Then for any countable sequence of subsets $A_1, A_2, \ldots$ of V, where $A_i \in \mathcal{U}$ for all i, the intersections $\bigcap_{i=1}^{\infty} A_i$ is in $\mathcal{U}$.*

Proof. Let $B_0 = V$, $B_1 = A_1$, $B_2 = A_1 \cap A_2, \ldots, B_n = A_1 \cap A_2 \cap \cdots \cap A_n, \ldots$, and $C_1 = B_0 - B_1$, $C_2 = B_1 - B_2, \ldots, C_n = B_{n-1} - B_n, \ldots$. Let $C_0 = \bigcap_{i=0}^{\infty} A_i$. Clearly $C_i \neq C_j$ for all $0 \leqslant i, j$, and $C_i \notin \mathcal{U}$ for all $i \geqslant 1$. Moreover, $\bigcup_{i=0}^{\infty} C_i = V$, and $B_i \in \mathcal{U}$ for all i, $i = 0, 1, 2, \ldots$. Suppose that $C_0 \notin \mathcal{U}$. If the members of each coalition C_j vote in block, it then follows as a consequence of Proposition A1 that one coalition, C_i, must dictate. It then easily follows that $C_i \in \mathcal{U}$. Consequently $C_i \cap B_i = \varnothing$ must be in $\mathcal{U}$, a contradiction.

Proposition A4. *If V is of measurable cardinality and $\mathcal{A}$ is countably infinite, then there exists a nondictatorial, Pareto, coalitionally-cheatproof* SCF *$F: \Sigma_1{}^V \to \mathcal{A}$.*

Proof. Since V is of measurable cardinality, there exists a nonprincipal ultrafilter, $\mathcal{U}$, over V such that for any countable sequence of sets $A_1, A_2, \ldots, \in \mathcal{U}$, the intersection $\bigcap_{i=1}^{\infty} A_i$ is also in $\mathcal{U}$. It is known (see [3, p. 182, Proposition 4.2.7]) that this implies that for any set $B = \{B_t\}_{t \in R}$ of subsets B_t of V, where the index set R is of cardinality $2^{\aleph_0}$ and each $B_t \in \mathcal{U}$, the intersection $\bigcap_{t \in R} B_t$ is then also in $\mathcal{U}$. It therefore follows that for any set $\{C_t\}_{t \in R}$ of subsets of $\mathcal{U}$, if $\bigcup_{t \in R} C_t = V$ and $C_{t_1} \cap C_{t_2} = \varnothing$

for $t_1 \neq t_2$ then for some $t_0 \in R$, $C_{t_0} \in \mathscr{U}$. Now given any $p = \{p_i\}_{i \in V} \in \Sigma_1{}^V$ and any $\bar{P} \in \Sigma_1$, let $Q_{\bar{P}} = \{i \mid i \in V \text{ and } p_i = \bar{P}\}$. Since there are only $2^{\aleph_0}$ elements in Σ_1, there are only $2^{\aleph_0}$ possible $Q_{\bar{P}}$'s. For each $P', P'' \in \Sigma_1$, if $P' \neq P''$ then $Q_{P'} \cap Q_{P''} = \varnothing$. Consequently for some $\hat{P} \in \Sigma_1$, $Q_{\hat{P}} \in \mathscr{U}$. We then set $F(p = \{p_i\}_{i \in V})$ equal to the maximal element of that $\hat{P}$. We do the same for any $p \in \Sigma_1{}^V$. Formally, $F(p = \{p_i\}_{i \in V}) =^{\text{def}}$ the maximal element of that $\hat{P} \in \Sigma_1$ such that $\{i \mid i \in V \text{ and } p_i = \hat{P}\} \in \mathscr{U}$. It is easily verified that the function $F: \Sigma_1{}^V \to \mathcal{A}$ constitutes a nondictatorial, Pareto, coalitionally-cheatproof SCF.

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Decentralized Resource Allocation and Increasing Returns

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We study the possibility of designing a decentralized resource allocation mechanism that yields optimal solutions in economies with increasing returns. After a brief review of the mechanisms already designed to cope with nonconvexities, we find out that either they fail to provide efficient solutions in Koopmans' sense, or else they use extremely complex messages. The question arises whether it is possible to define a nonwasteful and privacy-preserving mechanism over a class of environments with increasing returns using finite-dimensional vectors as messages. We show that the answer is no.

1. INTRODUCTION

It is well known that the competitive mechanism does not yield optimal solutions in the presence of externalities, public goods, increasing returns, or indivisibilities. This has given rise to a search for alternative mechanisms of resource allocation. The problem of designing satisfactory mechanisms for those classes of environments over which the competitive mechanism fails to perform satisfactorily is one of the most important subjects in the recent literature on economic planning. In this paper we are primarily concerned with the problem of optimal resource allocation in economies presenting certain types of increasing returns phenomena.

Given an optimality criterion, the problem of optimal resource allocation may be thought of as a mathematical programming problem where some objective function is maximized subject to the technological and resource availability constraints.[1] Yet the problem of designing a resource allocation mechanism cannot be reduced to the formulation and solution of an extremal problem. This is due, among other things, to the fact that the knowledge

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[1] The objective function is vector valued when either Koopmans' productive efficiency or Pareto-optimality criteria are used.

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about the economic environment is dispersed throughout the economy. Hence, we shall concentrate on a particular class of mechanisms, which are called informationally decentralized, satisfying certain conditions which guarantee that the initial dispersion of information and the limitations of communication are properly taken into account.

A resource allocation mechanism is formalized as an iterative process of information exchange in which all the agents send and receive messages. Once a stationary message is reached, an action is taken on the basis of the information conveyed by the stationary message. The iterative process of information exchange is described by the response functions, which express the message sent by the ith agent as a function of the previously received messages and the economic environment. A mechanism is said to be *privacy preserving* if the response function of the ith individual depends on the economic environment through the ith agent's characteristics only; that is, the response function of the ith agent is independent of other agents' characteristics. For instance, in the competitive mechanism the message received by an agent is a price vector and its response is the vector of proposed trades that maximizes his utility subject to the budget constraint defined by this price vector. Thus, its response is independent of other agents' utility functions and therefore the competitive mechanism is privacy preserving.

The privacy property takes into account the initial dispersion of information. This property, however, would be meaningless if there were no limitation on the amount of information which could be transmitted. A mechanism is said to be *point valued* if messages are points of some Euclidean space R^n. The mechanisms which satisfy the two properties mentioned above are called informationally decentralized.

Besides these informational properties, we want a satisfactory performance. A mechanism is said to be nonwasteful if every solution is optimal.

Several decentralized mechanisms have been designed to cope with increasing returns and other types of nonconvexities. While deferring a more detailed discussion, we may note here that these processes may be classified into two broad categories: the gradient processes (e.g., those in [3, 4, 8]) and the set-valued processes (e.g., the "greed process" in [12], the K process in [17], and the B process in [16]). The processes in the first category are unsatisfactory on the performance side because, as we shall see later, their solutions are not necessarily efficient in Koopmans' sense; i.e., they may be points of the interior of the feasible production set. The processes in the second category have some weakness on the informational side, since messages are sets of a rather complicated nature whose complete description may require infinitely many parameters.

The main purpose of this paper is to inquire whether it is possible to fill this gap by designing a mechanism that would meet the efficiency performance standard in nonconvex environments while retaining those informational attributes without which it could not be operated. In Section 5 we consider an economy with a single input, a single output, and two firms. One firm has a concave production function, while the other has a convex production function, so that there are increasing returns to scale. Theorem 1 states that there is no point-valued and privacy-preserving mechanism that is nonwasteful over a class of environments containing the one described above. It is interesting to note that the impossibility result obtains even when only a very mild type of nonconvexity is allowed. In Section 6 a similar result (Theorem 2) is shown to hold for an economy with a single consumer, a single producer, and two desired goods in which a quasi-concave utility function is to be maximized subject to a "nicely" nonconvex production possibility set. These results are related to that of Hurwicz [14], who shows that the informational requirements of a privacy-preserving process that is Pareto satisfactory over a class of piecewise linear environments are unbounded.

The results of this paper, although in the negative, are interesting for various reasons. First, they amount to an exploration of a certain area of what Mount and Reiter [23] have called "the set characterizing the possible trade-offs between environmental coverage and informational size." Second, the value of existing mechanisms emerges in a new light since, in some sense, their shortcomings in terms of performance, environmental coverage, or informational efficiency cannot be improved upon. Third, they bring into full daylight the extent to which increasing returns do make trouble and disqualify some naive policy recomendations.

To conclude this introduction, let us discuss briefly a particular methodological aspect of our technique of analysis. In showing impossibility theorems it is found useful to consider the simplest class of environments, since the theorem holds for any class containing it. The main theorems are consequently proven for very simple economies with only two agents.

2. DECENTRALIZED PLANNING PROCEDURES IN NONCONVEX ENVIRONMENTS

Given a planning procedure, it is important to know the characteristics of the economies in which it can be operated succesfully. We describe as environmental coverage of a planning procedure or resource allocation mechanism the class of such economies. Obviously, one would like the environmental coverage of a process to include those environments which might be encountered in practice.

Several decentralized planning procedures have been designed for economies satisfying the classical convexity assumptions. Some are price-guided processes, such as Lange's model of market socialism (see [21]) or the model of Arrow and Hurwicz [4]. Others are non-price-planning models such as those of Weitzman [25], Malinvaud [22], or Kornai and Lipták [20], among others. Unfortunately, there is considerable evidence that for many firms the production sets are nonconvex. In this paper we deliberately ignore the problems posed by the existence of externalities, public goods, and indivisibilities, in order to concentrate on the consequences of the relaxation of the convexity assumption. If we assume away convexity we allow all kinds of discontinuities and indivisibilities. However, we want to restrict our attention to what may be called the "shape" aspects of nonconvexities. Consequently we shall consider smooth and divisible environments so that the only nonclassical feature is the shape of the production sets.

In the recent literature the problem of constructing a decentralized planning routine for this class of nonconvex but otherwise well-behaved environments has been approached by using some variants of the gradient method. We can classify into this category the Arrow–Hurwicz modified Lagrangian process (see [4, Part IV, pp. 86–103]) and the planning procedures of Heal [8, 9], and Aoki [2]. In a recent book, Heal stated that "a planning routine that makes no use of prices can be shown to operate satisfactorily in the presence of increasing returns" (see [10, p. 155]). In order to clarify what can be rigorously claimed for this class of processes and to understand what is meant by "satisfactorily," let us make some general remarks which apply to all the models cited above.

These processes are based on the gradient method that climbs the nearest hill, not necessarily the highest. Hence only local optimality properties can be established in nonconvex environments. With the exception of the Arrow–Hurwicz modified Lagrangian process, not even local maxima can be guaranteed, since any critical point is an equilibrium of the process. If the initial point is not a critical point, then the process cannot converge to a local minimum but can converge to a point of inflection (see [15 pp. 13–14]).[2]

But there is a more fundamental point to be made. At first sight it seems natural to think that a local utility maximum, such as point B in Fig. 1, necessarily lies on the boundary of the feasible production set S. However, a closer examination reveals that this is not necessarily the case and that these processes may yield inefficient solutions in the sense that it is possible to produce more of each good. Indeed, as the following example shows, it may well happen that the point in the aggregate commodity space corresponding to the local utility maximizer (which lies in the n-fold Cartesian product of the commodity space) is an interior point of the feasible production possibility set.

Consider an economy with a single consumer and two desired goods, x and

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[2] For a rigorous study of the properties of Heal's process, see [11], where it is shown that the convergence to a critical point which is not a local maximum is unlikely in a well-defined sense.

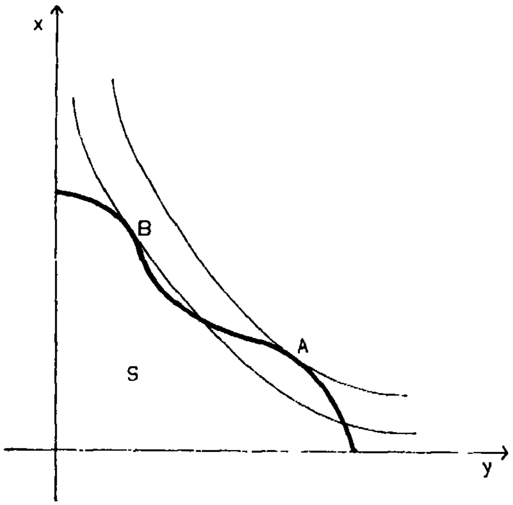

FIG. 1. Local utility maximum.

y, both produced by two firms with a fixed amount of primary goods. Let x_i and y_i denote the amounts produced by the ith firm. The utility function is given by $U(x, y) = xy$, where $x = x_1 + x_2$ and $y = y_1 + y_2$. The technology is described with the help of two functions given by

$$g^1(x, y_1) = y_1 - \max\{6 - 3x_1, (8 - x_1^2)^{1/2}\},$$
$$g^2(x_2, y_2) = x_2 + y_2 - 6.$$

The production sets of the firms are given by

$$Y^1 = \{(x_1, y_1) \in R^2 : g^1(x_1, y_1) \leq 0, x_1 \geq 0, y_1 \geq 0\},$$
$$Y^2 = \{(x_2, y_2) \in R^2 : g^2(x_2, y_2) \leq 0, x_2 \geq 0, y_2 \geq 0\},$$

and are depicted in Figs. 2 and 3, respectively.[3] The social feasible production set, $Y = Y^1 + Y^2$, is represented in Fig. 4. Now we claim that the point $(\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2) = (2, 3, 2, 3)$ is a local utility maximizer and is not efficient. In a neighborhood of this point the problem is

$$\max\,(x_1 + x_2)(y_1 + y_2)$$

---

[3] It might be objected that in this example some of the usual assumptions, such as factor mobility or the absence of joint production, are not satisfied. We have chosen it because it illustrates clearly the problems involved and lends itself to a simple graphical representation. However, it can be easily adapted to the case of firms producing a single output. In effect, suppose there are four firms. Firms 1 and 2 produce commodity x and firms 3 and 4 produce commodity y. On the other hand, firms 1 and 3 use input w only and firms 2 and 4 use input v only. Then firms 1 and 3 give rise to a production possibility set such as Y^1 in the example and firms 2 and 4 give rise to Y^2.

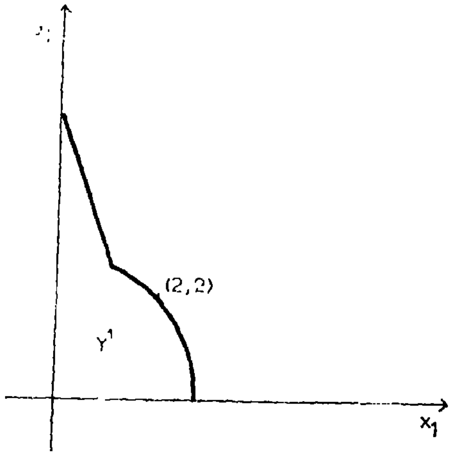

FIG. 2. Production possibilities of firm 1.

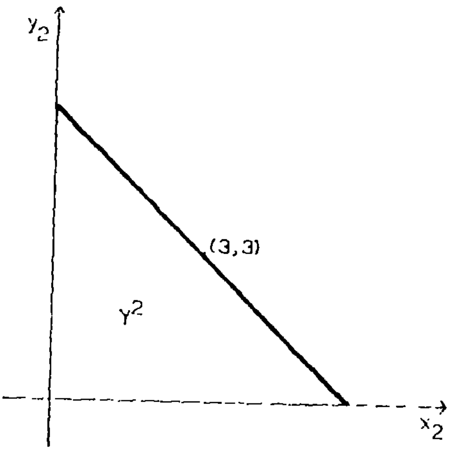

FIG. 3. Production possibilities of firm 2.

subject to

$$y_1 - (8 - x_1^2)^{1/2} = 0,$$
$$x_2 + y_2 - 6 = 0.$$

The Lagrangian expression for this problem is given by

$$L(x_1, x_2, y_1, y_2, \lambda_1, \lambda_2) = (x_1 + x_2)(y_1 + y_2) + \lambda_1[y_1 - (8 - x_1^2)^{1/2}] + \lambda_2(x_2 + y_2 - 6).$$

It is easily verified that for $\lambda_1 = -\frac{5}{4}$ and $\lambda_2 = -5$ all the partial derivatives of L evaluated at $(\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2)$ vanish. It is also easy to verify that the

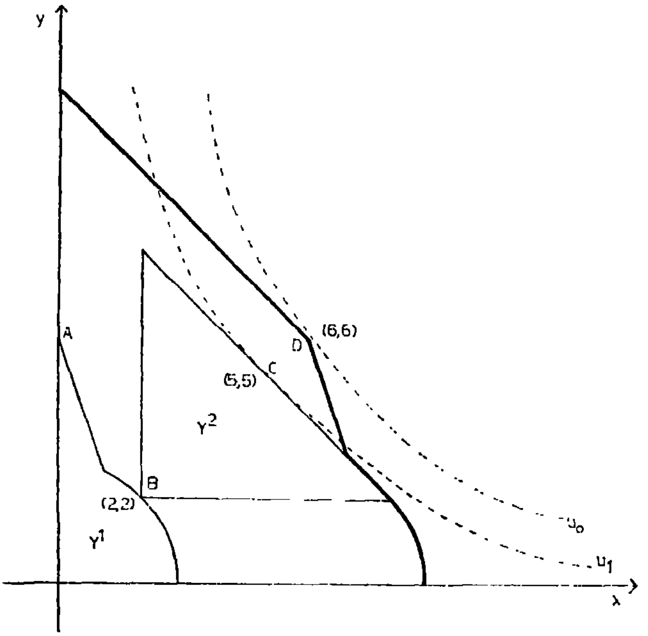

FIG. 4. An inefficient local utility maximizer.

constraint qualification and the second-order condition are satisfied at this point. Then by Burger's sufficiency theorem (see [5, Theorem 1], it follows that $(\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2) = (2, 3, 3, 3)$ is a local utility maximizer. However, the corresponding point in the aggregate space $(\bar{x}, \bar{y}) = (\bar{x}_1 + \bar{x}_2, \bar{y}_1 + \bar{y}_2) = (5, 5)$ is not efficient. For if each firm specializes in the production of a single commodity, we get $(\hat{x}_1, \hat{x}_2, \hat{y}_1, \hat{y}_2) = (0, 6, 6, 0)$ with a total production of $(\hat{x}, \hat{y}) = (6, 6)$ so that the production of each good is increased.

A graphical analysis may be helpful at this point. Points $(\bar{x}_1, \bar{y}_1) = (2, 2)$ and $(\bar{x}_2, \bar{y}_2) = (3, 3)$ are on the boundary of Y^1 and Y^2, respectively, as can be seen in Figs. 2 and 3. In Fig. 4 we show how the aggregate set $Y = Y^1 + Y^2$ is constructed by placing the origin of the set Y^2 on A and sliding it down following the boundary of Y^1 and keeping the axes parallel. Then Y consists of all the points covered by the sliding motion of Y^2. In Fig. 4 we represent the position of Y^2 when $(\bar{x}, \bar{y}) = (5, 5)$ is produced. It is clear that utility decreases with small movements around C or with small displacements of Y^2 around B and therefore $(\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2)$ maximizes utility locally. However, a "jump" of the origin of Y^2 from B to A allows the point D to be reached.

It is clear then that any process based on locating critical points is unable to prevent the obvious waste of resources in nonconvex environments. Are there mechanisms which allocate resources efficiently in nonconvex environments? The answer is in the affirmative; the "greed process" in [12] and the stochastic B process in [16] are well-known examples. The most striking feature of this class of set-valued processes is that, unlike the gradient-like processes, they are not based upon the traditional marginal conditions and therefore neither prices nor shadow prices enter the picture. They display an excellent static performance over a broad class of environments and, in the case of the B process, even convergence (in a probabilistic sense) can be established. Although they qualify as informationally decentralized according to Hurwicz's original definition (see [12, pp. 33–37]), their informational requirements are extremely burdensome: messages are subsets of the message space. The nature of messages is significantly simplified in the B process but, for the divisible case they consist of complex subsets whose complete description may require infinitely many parameters. Thus for practical purposes, it is reasonable to consider the set-valued processes as informationally infeasible.

Chipman [7] and Aoki [1] have approached the problem by imposing certain conditions upon the technology and the preference patterns of the consumer in order to ensure that the indifference surfaces are "more convex" than the transformation function. In particular, if the feasible production set is logarithmically convex and the utility function has a constant elasticity of substitution greater than 1, then the tangency conditions are necessary and sufficient for global optimality. This makes decentralization possible at the cost of a a severe loss in environmental coverage.

The discussion of the shortcomings of the existing mechanisms leads naturally to several conclusions as to the properties a process should satisfy to be considered satisfactory:

(a) The environmental coverage of the process should be sufficiently wide and at least include the class of all classical environments.

(b) On the informational side, the process should be privacy preserving, to take into account the initial dispersion of information, and use points of a finite-dimensional Euclidean space as messages.

(c) The process should lead to nonwasteful allocations.

In order to investigate the possibility of designing a process satisfying the above-mentioned properties, it is first necessary to set up a model.

3. THE MODEL

Let $\mathscr{I}$ denote the set of economic agents, $\mathscr{I} = \{1, 2, \ldots, n\}$ and let e^i denote the characteristic of the ith agent (e.g., initial endowments, preferences, and production set). Let E^i denote the topological space of all possible characteristics of the ith agent. Then the class of permissible environments is given by the product topological space $E = E^1 \times E^2 \times \cdots \times E^n$. Let $A = A^1 \times A^2 \times \cdots \times A^n$ denote the product topological space of joint actions (exchanges, production levels, etc.).

Intuitively speaking, a resource allocation process is a model describing how decisions are taken in a given economic system. A process is characterized by a *performance function* $\rho : E \rightarrow A$, indicating the action chosen for each possible economic environment. The image $\rho(e)$ of an environment $e \in E$ is called a *solution* of the process.

As we pointed out before, the knowledge about the economic environment is dispersed throughout the economy. Hence if consistent actions are to be taken some communication is necessary. The decision process is decomposed into two stages. In the first, an iterative exchange of formal messages takes place until an equilibrium state is reached. In the second, an action is taken on the basis of information conveyed by the equilibrium message. Formally we express this by saying that the performance function ρ has to be realized as the composition of a message correspondence[4] $\mu : E \rightarrow M$ and an outcome function $h : M \rightarrow A$, where M is the set of messages with which communication takes place.

$$\begin{tikzcd}
E \arrow[rr, "\rho"] \arrow[dr, "\mu"'] & & A \\
& M \arrow[ur, "h"'] &
\end{tikzcd} \qquad (1)$$

The problem of designing a resource allocation mechanism may be visualized in terms of the diagram displayed above. Instead of going from E to A directly we do it through M in two steps, which represent the two stages of the decision process. The design of a resource allocation process with a specified performance ρ consists in finding a correspondence μ, a message space M, and an outcome function h such that the triangular diagram (1) commutes.

The conceptual framework just outlined was first formulated by Hurwicz [12] and later was given the specific form displayed above by Mount and Reiter [23]. While referring the reader to these basic works for a complete analysis, let us present here a brief description of the basic notions.

Let $\mathcal{M}$ denote the set of signals (language) with which the exchange of information takes place and let $\mathcal{M}^{(n)}$ denote the n-fold Cartesian product of

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[4] By a correspondence $\mu : E \rightarrow M$ we mean a function with domain E and range in the power set of M such that for all $e \in E$, $\mu(e) \neq \emptyset$.

$\mathscr{M}$. The response functions $f^i : \mathscr{M}^{(n)} \times E \to \mathscr{M}$ define the difference equation system

$$m^i_{t+1} = f^i(m_t^1, m_t^2, \ldots, m_t^n; e), \qquad i \in \mathscr{I},$$

describing the message m^i_{t+1} sent by the ith agent at time $t + 1$ as a function of the messages received in the previous stage and the economic environment e. The process is in equilium when all the agents keep sending the same message over and over again, that is, $\bar{m} \in \mathscr{M}^{(n)}$ is an equilibrium message if $\bar{m}^i = f^i(\bar{m}; e)$ for all $i \in \mathscr{I}$. Let M be the set of all equilibrium messages given by

$$M = \{m \in \mathscr{M}^{(n)} : f(m; e) = m \text{ for some } e \in E\},$$

where $f = (f^1, f^2, \ldots, f^n)$. The outcome function associates a joint action to every equilibrium message. An adjustment process is defined by the ordered triple $[\mathscr{M}, f, h]$.

As we have seen, a resource allocation process is thought of as an adjustment process generating a set of equilibrium messages which, in a second stage, are translated into actions to be taken. Since we are not interested in its dynamic aspects, all we need to specify about the adjustment process is the set of equilibrium messages it generates for each economic environment and the outcome function. This is the approach taken by Mount and Reiter [23].

For each economic agent $i \in \mathscr{I}$, the correspondence (possibly multivalued) $\mu^i : E \to \mathscr{M}^{(n)}$, given by

$$\mu^i(e) = \{m = (m^1, m^2, \ldots, m^n) \in \mathscr{M}^{(n)} : f^i(m^1, m^2, \ldots, m^n; e) = m^i\}$$

indicates the messages in $\mathscr{M}^{(n)}$ that are stationary for the ith agent. Then the message correspondence $\mu : E \to \mathscr{M}^{(n)}$ is defined as

$$\mu(e) = \bigcap_{i=1}^{n} \mu^i(e), \tag{2}$$

or equivalently,

$$\mu(e) = \{m \in \mathscr{M}^{(n)} : f(m; e) = m\}, \tag{3}$$

and indicates the equilibrium messages corresponding to a given environment. The definition of μ as an intersection expresses the fact that an equilibrium message has to be stationary for every agent $i \in \mathscr{I}$. It is immediately seen that the message correspondence only takes values in $M \subseteq \mathscr{M}^{(n)}$. The outcome function $h : M \rightarrow A$ is defined as before. This completes the description of the concepts entering the fundamental triangular diagram (1).

It is clear now that the message correspondence describes the nature of the communication process. It is well known that there are encoding procedures by which a simple message can convey a lot of information. Hurwicz [13] provides some examples of such procedures related to the Peano curve. Following Mount and Reiter [23], in order to rule out inadmissible compression of information we impose the following smoothness condition on the message correspondence.

DEFINITION 1. Let E and M be topological spaces. The correspondence $\mu : E \dashrightarrow M$ is said to be *locally threaded* if and only if for every $e \in E$ there exist an open neighborhood N of e and a continuous function $s : N \rightarrow M$ such that for every $u \in N$, $s(u) \in \mu(u)$. The function s is called a *local thread* of μ.

Now we state a formal definition of the concept of the resource allocation process.

DEFINITION 2. Let E, M, and A be topological spaces, let $\mu : E \dashrightarrow M$ be a correspondence, and let $h : M \rightarrow A$ be a function. The ordered triple (M, μ, h) is said to be a *resource allocation process* on E if μ is a locally threaded correspondence and h is compatible with μ; i.e., if for every $e \in E$, if m and m' are in $\mu(e)$ then $h(m) = h(m')$.[5]

Note that the compatibility of μ and h guarantees that the performance of the process as characterized by $\rho : E \dashrightarrow A$, where $\rho = h \cdot \mu$, is indeed a function.

The problem of designing a process realizing a specified performance function $\rho : E \rightarrow A$ is to find an ordered triple $\Pi = (M, \mu, h)$ such that the mapping diagram (1) commutes. Without further conditions the problem is trivial, since any performance function can be realized by the process $\hat{\Pi} = (E, \zeta, \rho)$, where ζ denotes the identity function.

$$\begin{tikzcd}
E \arrow[rr, "\rho"] \arrow[dr, "\zeta"'] & & A \\
& \bar{E} \arrow[ur, "\rho"'] &
\end{tikzcd} \qquad (4)$$

---

[5] Unlike Mount and Reiter [23], we do not assume the outcome function h to be continuous. As Mount and Reiter [24] have shown, if h is required to be continuous the regularity condition imposed on the message correspondence is inherited in a stronger form by the performance function ρ and therefore the mechanism can only be expected to operate succesfully in environments in which the optimality correspondence Θ is sufficiently smooth. For our purposes this is an unnecessary restriction. The smoothness of the message correspondence is needed to rule out inadmissible compressions of information related to the Peano curve, but there is no reason to require h to be continuous

The conditions that we shall impose on the nature of the message space and the message correspondence try to reflect two basic facts: the initial dispersion of information and the limitations of communication. If the knowledge of the economic environment is dispersed throughout the economy so that, in particular, each agent knows only his own component of the environment and if all the information concerning other units' characteristics has to come via formal messages, then any informationally feasible process has to be such that the message sent by an agent $i \in \mathscr{I}$ depends upon the economic environment e through the ith characteristic e^i only. Hence the response function may be written as $m^i_{t+1} = f^i(m_t; e^i)$. Otherwise an economic agent would be supposed to send messages based on information he does not possess. A process satisfying this property is said to be *privacy preserving*. Obviously, if the process is privacy preserving the correspondence μ^i generated by the response function f^i is of the form $\mu^i : E^i \rightarrow M$; that is, the stationary messages of the ith agent depend on e^i alone and are independent of the characteristics of other agents. Consequently, the message correspondence satisfies the following property.

DEFINITION 3. Let $E = E^1 \times E^2 \times \cdots \times E^n$ be a product topological space. The correspondence $\mu : E \rightarrow M$ is said to be a *coordinate correspondence* if and only if for every $i \in \mathscr{I}$ there exists a correspondence $\mu^i : E^i \rightarrow M$ such that for every $e = (e^1, e^2, \ldots, e^n) \in E$,

$$\mu(e) = \bigcap_{i=1}^{n} \mu^i(e^i).$$

DEFINITION 4. A resource allocation process $\Pi = (M, \mu, h)$ is said to be *privacy preserving* if and only if μ is a coordinate correspondence.

The limitations of communication are reflected by requiring the message space to be a finite-dimensional Euclidean space. This condition tries to express the idea that messages have to be of reasonable complexity.

DEFINITION 5. A resource allocation process is said to be *point valued* if and only if messages are points of a finite-dimensional Euclidean space.

If the class of environments is sufficiently "rich," this requirement rules out the trivial process $\tilde{\Pi} = (E, \zeta, \rho)$ represented in (4). In general the message space is "smaller" than the class of environments and the difference between them could provide a mesure of the informational efficiency of a resource allocation process.

The privacy property, together with the limitation of the dimensions of the message space, is the basic feature characterizing informational decentralization.[6]

Besides these informational requirements, we want a satisfactory performance. In this paper we restrict our attention to the static optimality properties. The optimality correspondence $\Theta : E \twoheadrightarrow A$ associates a set of optimal actions to each economic environment. We shall require the process to yield optimal actions, that is, for every $e \in E$, $\rho(e) \in \Theta(e)$.

DEFINITION 6. Given a class of environments E and an optimality correspondence $\Theta : E \twoheadrightarrow A$, a process $\Pi = (M, \mu, h)$ is said to be *nonwasteful* over E if, for every $e \in E$ and every $m \in \mu(e)$, $h(m) \in \Theta(e)$.

4. PRELIMINARY PROPOSITIONS

Let E be a class of environments, let $\mu : E \twoheadrightarrow M$ be a message correspondence, and let $h : M \twoheadrightarrow A$ be an outcome function defining a process Π. Given a subclass of environments $\bar{E} \subseteq E$, we can define a process on $\bar{E}$ given by $\bar{\Pi} = (\bar{M}, \bar{\mu}, \bar{h})$, where $\bar{M} = \mu[\bar{E}]$, $\bar{\mu} = \mu \mid \bar{E}$ is the restriction of μ to $\bar{E}$ and $\bar{h} = h \mid \bar{M}$ is the restriction of h to $\bar{M}$. The following propositions show that the process $\bar{\Pi}$ thus defined inherits the smoothness and privacy properties of the original process Π.

PROPOSITION 1. *Let $\mu : E \twoheadrightarrow M$ be a coordinate correspondence and let $\bar{E} \subseteq E$. Then the restriction of μ to $\bar{E}$, denoted by $\bar{\mu} = \mu \mid \bar{E}$, is a coordinate correspondence.*

Proof. If μ is a coordinate correspondence, $\mu(e) = \bigcap_{i=1}^{n} \mu^i(e^i)$. Define $\bar{\mu}^i = \mu^i \mid \bar{E}^i$, where $\bar{E}^i$ is the ith projection of $\bar{E}$. Then for every $e \in \bar{E}$ we have

$$\bar{\mu}(e) = \mu(e) = \bigcap_{i=1}^{n} \mu^i(e^i) = \bigcap_{i=1}^{n} \bar{\mu}^i(e^i).$$

Hence, $\bar{\mu}$ is a coordinate correspondence. Q.E.D.

PROPOSITION 2. *Let $\mu : E \twoheadrightarrow M$ be a locally threaded correspondence and let $\bar{E} \subseteq E$. Then $\bar{\mu} = \mu \mid \bar{E}$ is locally threaded.*

Proof. By Definition 1, if μ is locally threaded, for every $e \in E$ there exist an open neighborhood N of e and a continuous function $s : N \twoheadrightarrow M$ such that $s(u) \in \mu(u)$ for all $u \in N$. Define $\bar{N} = N \cap \bar{E}$ and $\bar{s} = s \mid \bar{N}$. Then $\bar{s}$ is a local thread of $\bar{\mu}$. In effect, $\bar{N}$ is open in the relative topology of $\bar{E}$, $\bar{s}$ is continuous, and for each $u \in \bar{N}$ we have $\bar{\mu}(u) = \mu(u)$. Hence $\bar{s}(u) \in \bar{\mu}(u)$ for all $u \in \bar{N}$. Q.E.D.

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[6] Our concept of informational decentralization requires the message space to be finite-dimensional. Hurwicz [13] uses a stronger requirement restricting the message space to a Euclidean space having the same dimension as the commodity space.

DEFINITION 7. The correspondence $\mu : E \twoheadrightarrow M$ is said to be *injective* if for every $e \in E$ and $e' \in E$, $\mu(e) \cap \mu(e') \neq \phi$ implies $e = e'$.

PROPOSITION 3. *Let X be a subset of R^n with nonempty interior and let Y be a subset of R^m. Suppose that there exists a locally threaded injective correspondence $\mu : X \twoheadrightarrow Y$. Then the dimension of Y is at least n.*

Proof. Take a point x^* in the interior of X. Since μ is locally threaded, there exist an n-dimensional open neighborhood V of x^* and a continuous function $s : V \twoheadrightarrow Y$ such that $s(x) \in \mu(x)$ for all $x \in V$.

Now we claim that s is one-to-one. To show this, let $x \in V$ and $x' \in V$ be such that $x \neq x'$. Since μ is injective, $\mu(x) \cap \mu(x') = \phi$. On the other hand, by definition of s, we have $s(x) \in \mu(x)$ and $s(x') \in \mu(x')$. Thus $s(x) \neq s(x')$.

Since X is Euclidean we can take a compact set $\bar{V} \subseteq V$ with nonempty interior so that $\bar{V}$ is n-dimensional. Let $\bar{s}$ be the restriction of s to $\bar{V}$, and let $\bar{U} = s[\bar{V}]$. Since $\bar{s}$ is continuous and one-to-one, $\bar{V}$ compact and $\bar{U}$ Hausdorff, then $\bar{s}$ is a homeomorphism (see [18, Theorem 8, p. 141]). Hence it follows that we can embed an n-dimensional disc into the Euclidean set Y and therefore the dimension of Y is at least n. Q.E.D.

Now we shall show the Injectiveness Lemma, which tries to characterize a subclass of environments with the property that each member of the subclass needs different equilibrium messages if the process is to be nonwasteful. Hence the message space should contain enough information so as to distinguish between members of the subclass. Then, if some smoothness conditions are fulfilled, the dimension of this subclass puts a lower bound on the dimensional requirements of the process.

We restrict our attention to environments with only two agents, although the results could be extended to a more general case.

DEFINITION 8. Let $E = E^1 \times E^2$ be a class of environments and $\Theta : E \twoheadrightarrow A$ an optimality correspondence. A subset $\bar{E}$ of E is said to possess the *uniqueness property* if and only if for every $\tilde{e} = (\tilde{e}^1, \tilde{e}^2) \in \bar{E}$ and every $\bar{e} = (\bar{e}^1, \bar{e}^2) \in \bar{E}$, $\Theta(\bar{e}^1, \bar{e}^2) \cap \Theta(\bar{e}^1, \tilde{e}^2) \cap \Theta(\tilde{e}^1, \bar{e}^2) \cap \Theta(\tilde{e}^1, \tilde{e}^2) \neq \phi$ implies $\bar{e} = \tilde{e}$.

INJECTIVENESS LEMMA.[7] *Let $\Pi = (M, \mu, h)$ be a privacy-preserving process that is nonwasteful over a class of environments $E = E^1 \times E^2$. Let $\bar{E} \subseteq E$ be a*

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[7] The idea in this lemma is similar to that of the Single-Valuedness Lemma in [13].

subclass of environments with the uniqueness property. Then the restriction of μ *to* $\bar{\mathbf{E}}$*, denoted by* $\bar{\mu}$*, is an injective correspondence.*

Proof. We have to show that given two arbitrary elements of $\bar{E}$, $\bar{e}$ and $\tilde{e}$, $\bar{\mu}(\bar{e}) \cap \bar{\mu}(\tilde{e}) \neq \phi$ implies $\bar{e} = \tilde{e}$. Since the process Π is privacy preserving, the message correspondence μ is a coordinate correspondence. By Proposition 1 it follows that $\bar{\mu}$ is also a coordinate correspondence. Take $m \in \bar{\mu}(\tilde{e}) \cap \bar{\mu}(\bar{e})$. Since $\bar{\mu}$ is a coordinate we have that $m \in \bar{\mu}^1(\bar{e}^1) \cap \bar{\mu}^1(\tilde{e}^1) \cap \bar{\mu}^2(\bar{e}^2) \cap \bar{\mu}^2(\tilde{e}^2)$ and this implies $m \in \bar{\mu}(\tilde{e}^1, \tilde{e}^2) \cap \bar{\mu}(\tilde{e}^1, \bar{e}^2) \cap \bar{\mu}(\bar{e}^1, \tilde{e}^2) \cap \bar{\mu}(\bar{e}^1, \bar{e}^2)$. Let $a = h(m)$. Since the process Π is nonwasteful it follows that $a \in \Theta(\tilde{e}^1, \tilde{e}^2) \cap \Theta(\tilde{e}^1, \bar{e}^2) \cap \Theta(\bar{e}^1, \tilde{e}^2) \cap \Theta(\bar{e}^1, \bar{e}^2)$. But the fact that this intersection is nonempty, together with the fact that $\bar{E}$ has the uniqueness property, implies that $\bar{e} = \tilde{e}$. Q.E.D.

5. POSSIBILITY OF DECENTRALIZATION IN PRODUCTION

In this section we show that any decentralized planning process which uses points of a finite-dimensional space as messages is unable to prevent the obvious waste of economic resources. The structure of the proof is as follows. First we construct a subclass of environments with the uniqueness property. Then by the Injectiveness Lemma it follows that every environment in this subclass needs different messages so that, intuitively speaking, the message space has to be at least as "big" as the subclass of environments. Since the dimension of the subclass constructed can be made as large as desired the result follows.

Consider a production economy with a single produced good y, a single primary good x, and two firms, $i \in \{1, 2\}$, with production functions $y^i = \Psi^i(x^i)$, where y^i and x^i denote the amounts respectively produced and used by the ith firm. We assume that the production functions of both firms are increasing, twice continuously differentiable, and go through the origin. Firm 2 is known to have a classical concave production function, but firm 1 presents increasing returns, since its production function is supposed to be convex. It is important to note that this is a specially nice class of nonconvex environments, since potential sources of trouble such as indivisibilities and kinked or wiggling functions are ruled out.

The total initial endowment of the primary good is known to be $w = 1$. Let $I = [0, 1]$ denote the closed unit interval. The input allocation among the two firms may be described by a single parameter $t \in I$, where $x^1 = t$ and $x^2 = 1 - t$. It is convenient to define the production functions in terms of t alone, so that

$$e_1(t) = \Psi^1(t),$$
$$e_2(t) = \Psi^2(1 - t),$$

and total output is a function of t, $y = e_1(t) + e_2(t)$. Let e_i' and e_i'' denote the first and second derivatives of the function $e_i(t)$. Then the class of environments is given by $E = E^1 \times E^2$, where

$$E^1 = \{e_1 \in C^2(I): e_1(0) = 0,\ e_1' > 0, \text{ and } e_1'' > 0\}, \tag{5}$$

$$E^2 = \{e_2 \in C^2(I): e_2(1) = 0,\ e_2' < 0, \text{ and } e_2'' < 0\}, \tag{6}$$

and $C^2(I)$ denotes the topological space of all twice continuously differentiable functions defined on the compact interval I with the C^2 norm topology.

The natural optimality criterion for this economy is the maximization of total output so that the optimality correspondence $\Theta : E \rightarrow I$ is given by

$$\Theta(e_1, e_2) = \{\bar{t} \in I : e_1(\bar{t}) + e_2(\bar{t}) \geq e_1(t) + e_2(t) \text{ for all } t \in I\}. \tag{7}$$

Let $P(I)$ denote the set of all polynomial functions defined on I of the form

$$e_i(t) = \sum_{n=0}^{N} e_{in} t^n,$$

where N is a positive integer and $e_{in} \in R$. Then define

$$E_N^1 = \left\{ e_1 \in P(I): e_1(t) = \sum_{n=1}^{N} e_{1n} t^n \text{ and } e_{1n} \geq 0 \text{ for } n = 1,\dots, N \right\},$$

$$E_N^2 = \left\{ e_2 \in P(I): e_2(t) = \sum_{n=1}^{N} e_{2n} - \sum_{n=1}^{N} e_{2n} t^n \text{ and } e_{2n} \geq 0 \text{ for all } n \right\},$$

$$E_N = \left\{ (e_1, e_2) \in E_N^1 \times E_N^2: \sum_{n=1}^{N} e_{1n} = 1 \text{ and } e_{1n} = e_{2n} \text{ for all } n \right\}.$$

It is easily verified that E_N is a subclass of the class of environments $E = E^1 \times E^2$ given in (5) and (6).

Proposition 4. *The subclass of environments E_N has the uniqueness property.*

Proof. Let $\tilde{e} = (\tilde{e}_1, \tilde{e}_2)$ and $\bar{e} = (\bar{e}_1, \bar{e}_2)$ be two points in E_N such that $\Theta(\tilde{e}_1, \tilde{e}_2) \cap \Theta(\tilde{e}_1, \bar{e}_2) \cap \Theta(\bar{e}_1, \tilde{e}_2) \cap \Theta(\bar{e}_1, \bar{e}_2) \neq \phi$. We have to show that this implies $\bar{e} = \tilde{e}$. Since both points belong to E_N we have that $\bar{e}_{1n} = \bar{e}_{2n}$ and $\tilde{e}_{1n} = \tilde{e}_{2n}$ for all $n \in \{1, 2,\dots, N\}$. Since the intersection is nonempty, take $\bar{t} \in \Theta(\bar{e}_1, \tilde{e}_2) \cap \Theta(\tilde{e}_1, \bar{e}_2)$. By definition of Θ in (7) it follows that $\sum_{n=1}^{N} \bar{e}_{1n}\bar{t}^n + \sum_{n=1}^{N} \tilde{e}_{2n} - \sum_{n=1}^{N} \tilde{e}_{2n}\bar{t}^n \geqslant \sum_{n=1}^{N} \bar{e}_{1n}t^n + \sum_{n=1}^{N} \tilde{e}_{2n} - \sum_{n=1}^{N} \tilde{e}_{2n}t^n$ for all $t \in I$. Canceling the second term in both sides and factoring out $\bar{t}^n$ and t^n,

$$\sum_{n=1}^{N} (\bar{e}_{1n} - \tilde{e}_{1n})\, \bar{t}^n \geq \sum_{n=1}^{N} (\bar{e}_{1n} - \tilde{e}_{1n})\, t^n \qquad \text{for all} \quad t \in I. \tag{8}$$

In a similar fashion, we can establish

$$\sum_{n=1}^{N} (\tilde{e}_{1n} - \bar{e}_{1n})\, \bar{t}^{n} = \sum_{n=1}^{N} (\tilde{e}_{1n} - \bar{e}_{1n})\, t^{n} \qquad \text{for all} \quad t \in I. \tag{9}$$

Since (8) and (9) hold in particular for $t = 0$, we get the equality $\sum_{n=1}^{N} (\bar{e}_{1n} - \tilde{e}_{1n})\, \bar{t}^{n} = 0$. Using this fact, together with (8) and (9), we establish $\sum_{n=1} (\bar{e}_{1n} - \tilde{e}_{1n})\, t^{n} = 0$ for all $t \in I$. If a polynomial function is identically zero over the interval I, it follows that it is the zero polynomial function so that $\tilde{e}_{1n} = \bar{e}_{1n}$ for all $n \in \{1, 2, \ldots, N\}$. Hence, $\bar{e}_1 = \tilde{e}_1$. On the other hand, by definition of E_N we have $\bar{e}_{2n} = \tilde{e}_{2n}$ for all n. Thus $\bar{e}_2 = \tilde{e}_2$, and therefore $\bar{e} = \tilde{e}$. Q.E.D.

THEOREM 1. *There does not exist a point-valued and privacy-preserving process that is nonwasteful over a class of environments containing the class* $E = E^1 \times E^2$ *defined in* (5) *and* (6).

Proof. Suppose there existed such a process and let it be $\Pi = (M, \mu, h)$. Since Π is privacy preserving and point valued, μ is a locally threaded and coordinate correspondence, and M is a subset of a finite-dimensional Euclidean space. Let Q be the dimension of M.

Consider the class of environments E_N for $N = Q + 1$. By Proposition 4 we know that E_N has the uniqueness property. Since Π is privacy preserving we can apply the Injectiveness Lemma and conclude that the restriction of μ to E_N is an injective correspondence. On the other hand, it is easily verified that E_N is topologically equivalent to the $(N - 1)$-dimensional simplex. Since the restriction of μ to E_N is locally threaded and injective (by Proposition 2), it follows by Proposition 3 that the dimension of M is at least $N - 1$. Hence $Q \geq N - 1$ and this contradicts our choice of N. Q.E.D.

Let us discuss briefly the scope of this result. Consider the simple production economy described at the beginning of this section. If both firms have strictly concave production functions, an input allocation is optimal if and only if it is a critical point (i.e., if the first-order conditions for a maximum are met at this point). Taking advantage of this static characterization of maxima it is possible to design a gradient-like adjustment process which in terms of our formal model will be characterized by a three-dimensional Euclidean space as a message space, the correspondence $\mu : E \to M$ given by $\mu(\Psi^1, \Psi^2) = \{(x_1, x_2, p) \in R_+^3 : \Psi_x^1(x_1) = \Psi_x^2(x_2) = p \text{ and } x_1 + x_2 = w\}$ (where Ψ_x^i denotes the derivative of Ψ^i) and the outcome function h given by $h(x_1, x_2, p) = (x_1, x_2)$. In this case messages have a familiar interpretation in terms of prices and proposed allocations. Hence if production sets are strictly convex we can use smooth search procedures to locate optimal points.

What happens if all firms have strictly convex production functions? It turns out that in this case decentralization is possible even though all firms display increasing returns to scale. For this class of environments a necessary condition for optimality is that the total amount of a given input has to be allocated to a single firm, Hence, the search procedure can be safely restricted to the extreme points of the set of feasible input allocations. Since the number of such points is finite, the optimal allocation may be determined by comparing total output at every point. Taking advantage of these facts, the *C* process[8] guarantees global optimality and is decentralized. Let us illustrate the working of this process for the case of a single input, a single output, and two firms. If both production function are strictly convex there are only two input allocations as potential maximizers: $(x_1, x_2) = (0, w)$ and $(x_1, x_2) = (w, 0)$. Hence, the only information the center needs is the output level at $x_i = w$ for every i. The message correspondence is then given by $\mu(\Psi^1, \Psi^2) = [\Psi^1(w), \Psi^2(w)]$. The outcome function, which translates equilibrium messages into actions, is given by

$$\begin{aligned} h[\Psi^1(w), \Psi^2(w)] &= (0, w) \qquad \text{if } \Psi^1(w) \leq \Psi^2(w), \\ &= (w, 0) \qquad \text{otherwise.} \end{aligned}$$

The process may be generalized to the case of a single output, n inputs, and m firms. Three features of the *C* process are worth noticing. First, the dimension of the message space is an exponential function of the number of inputs. Second, messages do not bear any relationship with prices, input allocations, or marginal productivities. Third, the search procedure is discontinuous in the sense that it jumps from one extreme point to another.

Two assumptions about the class of economic environments over which the *C* process performs satisfactorily seem to be too restrictive for the process to have any interest other than academic. First, all the firms in the economy have production functions with the wrong kind of curvature. Second, there is a single produced good. If we try to relax the first assumption and admit the possibility of some firms with concave production functions, then Theorem 1 applies and decentralization is impossible. If we try to relax the second assumption and consider more than one produced good, it is necessary to introduce a utility function. The problem is then to maximize a quasi-concave utility function subject to a possibly nonconvex feasible production set. In the next section we investigate the possibility of solving this problem by means of a decentralized procedure.

6. POSSIBILITY OF DECENTRALIZATION WITH CONSUMPTION

Consider an economy with two desired goods, to be denoted by x and y; and two agents, a producer and a consumer. The producer is characterized by a function g defining the feasible set. The consumer is characterized by a utility function u.

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[8] *The C process and the optimality conditions are fully developed in [6].*

The class E^2 of possible characteristics of the producer is the set of all feasible production sets defined by a function $g \in C^2(I)$ such that $g(0) = 1$, $g(1) = 0$, $g' < 0$, and $g'' \geq 0$. On the other hand, the class E^1 of possible characteristics of the consumer is the space of all utility functions defined in the two-dimensional positive orthant R_+^2 that are quasi-concave and have positive partial derivatives everywhere. Formally,

$$E^1 = \{u \in C^2(R_+^2) : \partial u/\partial x > 0,\ \partial u/\partial y > 0, \text{ and } u \text{ is quasi-concave}\}, \tag{10}$$

$$E^2 = \{ g \in C^2(I) : g(0) = 1,\ g(1) = 0,\ g' < 0, \text{ and } g'' \geq 0\}, \tag{11}$$

where I denotes the closed unit interval and $C^2(I)$ is the topological space of all twice continuously differentiable functions defined on I with the C^2 norm topology. The class of environments is then given by $E = E^1 \times E^2$ and the typical element of E is of the form $e = (u, g) \in E$. We assume that once the production level of x has been determined, the production level of y is automatically set equal to $y = g(x)$ by the producer. Thus, the action space is the closed unit interval I. The natural optimality criterion is the maximization of the utility function subject to the feasible set so that the optimality correspondence $\Theta : E \to I$ is given by $\Theta(u, g) = \{\bar{x} \in I : u(\bar{x}, g(\bar{x})) \geq u(x, g(x))$ for all $x \in I\}$.

The structure of the proof of Theorem 2 is analogous to that of Theorem 1. We begin by defining a subclass of environments with the uniqueness property. Let $\bar{E}^1 = \{u \in E^1 : u(\lambda x, \lambda y) = \lambda u(x, y)$ for all $\lambda > 0$ and $(x, y) \in R_+^2\}$ and $\bar{E}_N^2 = \{ g \in E^2 : g(x) = \sum_{n=1}^{N} a_n(1 - x)^n$, where $a_n \in R$, $a_n \geq 0$ and $\sum_{n=1}^{N} a_n = 1\}$. Then we define a subclass of environments $\bar{E}_N \subseteq E$: $\bar{E}_N = \{(u, g) \in \bar{E}^1 \times \bar{E}_N^2 : u(x, g(x)) = K$ for all $x \in I\}$, where K is a fixed real number.

Proposition 5. *$\bar{E}_N$ has the uniqueness property.*

Proof. Let $(\bar{u}, \bar{g}) \in \bar{E}_N$, $(\tilde{u}, \tilde{g}) \in \bar{E}_N$, and suppose that the following holds: $\Theta(\bar{u}, \bar{g}) \cap (\bar{u}, \tilde{g}) \cap \Theta(\tilde{u}, \bar{g}) \cap \Theta(\tilde{u}, \tilde{g}) \neq \varnothing$. We have to show that this implies $(\bar{u}, \bar{g}) = (\tilde{u}, \tilde{g})$. By definition of $\bar{E}_N$, we have

$$\bar{g}(0) = \tilde{g}(0) = 1, \tag{12}$$

$$\bar{u}(x, \bar{g}(x)) = K \qquad \text{for all } x \in I, \tag{13}$$

$$\tilde{u}(x, \tilde{g}(x)) = K \qquad \text{for all } x \in I. \tag{14}$$

Moreover, since for this class of environments the utility function is homogeneous and has positive partial derivatives everywhere, we have

$$y > y' \Leftrightarrow u(x, y) > u(x, y'). \tag{15}$$

Take $\bar{x} \in \Theta(\bar{u}, \tilde{g}) \cap \Theta(\tilde{u}, \bar{g})$. By definition of the optimality correspondence it follows that $\bar{u}(\bar{x}, \tilde{g}(\bar{x})) \geq \bar{u}(x, \tilde{g}(x))$ and $\tilde{u}(\bar{x}, \bar{g}(\bar{x})) \geq \tilde{u}(x, \bar{g}(x))$ for all $x \in I$. Since these inequalities hold in particular for $x = 0$, using (12), (13), and (14) we get

$$\bar{u}(\bar{x}, \tilde{g}(\bar{x})) \geq \bar{u}(0, \tilde{g}(0)) = \bar{u}(0, \bar{g}(0)) = K, \tag{16}$$

$$\tilde{u}(\bar{x}, \bar{g}(\bar{x})) \geq \tilde{u}(0, \bar{g}(0)) = \tilde{u}(0, \tilde{g}(0)) = K. \tag{17}$$

Step 1. We prove that $\tilde{u}(\bar{x}, \bar{g}(\bar{x})) = K$.

By (17) we know that $\tilde{u}(\bar{x}, \bar{g}(\bar{x})) \geq K$. Suppose that $\tilde{u}(\bar{x}, \bar{g}(\bar{x})) > K$. Since (14) holds in particular for $x = \bar{x}$, we have $\tilde{u}(\bar{x}, \bar{g}(\bar{x})) > \tilde{u}(\bar{x}, \tilde{g}(\bar{x}))$. By (15) it follows that $\tilde{g}(\bar{x}) < \bar{g}(\bar{x})$. Using (15) again we get $\bar{u}(\bar{x}, \tilde{g}(\bar{x})) < \bar{u}(\bar{x}, \bar{g}(\bar{x})) = K$ and this contradicts (16), establishing our claim.

Step 2. We show that $\tilde{g} = \bar{g}$.

Suppose not, so that for some $\tilde{x} \in I$, $\tilde{g}(\tilde{x}) \neq \bar{g}(\tilde{x})$. Let us assume without loss of generality that $\bar{g}(\tilde{x}) > \tilde{g}(\tilde{x})$. Then, by (15) it follows that $\tilde{u}(\tilde{x}, \bar{g}(\tilde{x})) > \tilde{u}(\tilde{x}, \tilde{g}(\tilde{x}))$. From (14), taking $x = \tilde{x}$, and the result of step 1, $\tilde{u}(\tilde{x}, \tilde{g}(\tilde{x})) = \tilde{u}(\bar{x}, \bar{g}(\bar{x}))$. Hence $\tilde{u}(\tilde{x}, \bar{g}(\tilde{x})) > \tilde{u}(\bar{x}, \bar{g}(\bar{x}))$, and this contradicts the optimality of $\bar{x}$ since we have assumed $\bar{x} \in \Theta(\tilde{u}, \bar{g})$.

Step 3. We finally show that $\bar{u} = \tilde{u}$.

It is clear that our assumptions on g guarantee that the projecting cone of the graph of g from the origin is the nonnegative orthant R_+^2. Hence, given any $(x, y) \in R_+^2$ there exist $\lambda > 0$ and $\tilde{x} \in I$ such that $x = \lambda\tilde{x}$ and $y = \lambda\bar{g}(\tilde{x}) = \lambda\tilde{g}(\tilde{x})$. On the other hand, by definition of $\bar{E}^1$, we know that $\bar{u}$ and $\tilde{u}$ are homogeneous of degree one. Thus, given any $(x, y) \in R_+^2$, we have $\bar{u}(x, y) = \bar{u}(\lambda\tilde{x}, \lambda\tilde{g}(\tilde{x})) = \lambda K = \lambda\tilde{u}(\tilde{x}, \bar{g}(\tilde{x})) = \tilde{u}(\lambda\tilde{x}, \lambda\bar{g}(\tilde{x})) = \tilde{u}(x, y)$, where use has been made of (13) and (14) for $x = \tilde{x}$. Q.E.D.

THEOREM 2. *There does not exist a point-valued and privacy-preserving process that is nonwasteful over a class of environments containing the class* $E = E^1 \times E^2$ *defined in* (10) *and* (11).

Proof. We have defined a subclass of environments $\bar{E}_N$ with the uniqueness property that is homeomorphic to the $(N - 1)$-dimensional simplex. The proof is then analogous to that of Theorem 1. Q.E.D.

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Houthakker's Axiom in the Theory of Rational Choice*

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1. INTRODUCTION

In his celebrated classic paper [4], Houthakker strengthened Samuelson's weak axiom of revealed preference [7, 8] into what he called *semitransitivity*, and showed that the Lipschitz-continuous demand function of a competitive consumer satisfying his axiom does possess a generating utility function.[1] Uzawa [11], Arrow [1], and others extended the conceptual framework of revealed preference theory so as to make it applicable to a wider class of problems. Instead of confining our attention to a demand function of a competitive consumer, we are now concerned with a *choice function* over a family of nonempty subsets of a basic nonempty set. A natural question suggests itself: What property of a choice function guarantees the existence of a generating preference ordering (GPO)? In particular, does Houthakker's axiom, if suitably reformulated, qualify as such? It is this problem of the existence of a GPO that constitutes the problem of *rationalizability* of a choice function, which is a choice-functional counterpart of the integrability problem in demand theory.

In the literature we have two answers to this question, depending on the extent of the domain of a choice function. On the one hand, if the family over which a choice function is defined contains all finite subsets of the whole space, the weak axiom ensures the existence of a GPO and that the *strong axiom of revealed preference* (which was so named by Samuelson [8] and attributed to Houthakker) is equivalent to the weak axiom [1, 9]. On the other hand, if we do not impose any such additional assumption on the domain, the strong axiom is necessary but not sufficient for the existence of a GPO [3, 10].[2] Richter [5] and Hansson [3] proposed in this general setting a

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[1] Further results on this problem are found in papers collected in [2]. See, especially, Uzawa [2, Chap. 1] and Hurwicz-Richter [2, Chap. 3].

[2] This statement is true for one version of the strong axiom, which is used by Arrow [1], Hansson [3], Sen [9] and Suzumura [10]. More about this in the final section.

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necessary and sufficient condition for the existence of a GPO, which was called the *congruence axiom* by Richter.

The purpose of this paper is to show that Houthakker's semitransitivity axiom, if suitably formalized in the choice-functional context, is in fact necessary and sufficient for the existence of a GPO. Therefore it follows that, contrary to the prevailing interpretation, the strong axiom is not a legitimate formalization of Houthakker's axiom which is *equivalent* to the congruence axiom. In other words, Houthakker's axiom, unlike the strong axiom, provides us with the precise restriction on a choice function for the rationalizability thereof, just as it provided us with the precise restriction on a Lipschitz-continuous demand function for the integrability thereof.

2. RATIONALIZABILITY

2.1. Let X be a nonempty set which stands for a fixed universe of alternatives. We assume that there is a well-specified family K of nonempty subsets of X. The pair (X, K) will be called a *choice space*. A *choice function* on a choice space (X, K) is a function C defined on K which assigns a nonempty subset (*choice set*) $C(S)$ of S to each $S \in K$.

A *preference relation* R on X is a binary relation on X, namely a subset of a Cartesian product $X \times X$. Associated with a given preference relation R, an infinite sequence of binary relations $\{R^{(\tau)}\}_{\tau=1}^{\infty}$ is defined by $R^{(1)} = R$, $R^{(\tau)} = \{(x, y): (x, z) \in R^{(\tau-1)} \;\&\; (z, y) \in R \text{ for some } z \in X\}$ $(\tau \geqslant 2)$. The *transitive closure* of R is then defined by $T(R) = \bigcup_{\tau=1}^{\infty} R^{(\tau)}$.[3] The *strict preference relation* P_R corresponding to a preference relation R is an asymmetric component of R:

$$P_R = \{(x, y): (x, y) \in R \;\&\; (y, x) \notin R\}. \tag{1}$$

We say that a preference relation R is *transitive* if $(x, y) \in R$ and $(y, z) \in R$ imply $(x, z) \in R$, *acyclic* if $(x, x) \notin T(P_R)$, and *total* if either $(x, y) \in R$ or $(y, x) \in R$ for all x and y in X. R is said to be an *ordering* if it is transitive and total. For every $S \in K$, we define

$$G(S, R) = \{x: x \in S \;\&\; (x, y) \in R \text{ for all } y \in S\}, \tag{2}$$

which is the set of all R greatest points in S.

2.2. A preference relation R on X is said to rationalize a choice function C on (X, K) if we have

$$C(S) = G(S, R) \qquad \text{for every} \quad S \in K. \tag{3}$$

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[3] It is easy to see that T satisfies the axiom of *closure operations*: $(\alpha) R \subset T(R)$ for every R, (β) $R \subset R'$ implies $T(R) \subset T(R')$ for every R and R', (γ) $T[T(R)] = T(R)$ for every R, and $(\delta) T(\varnothing) = \varnothing$.

A choice function C is said to be *rational* if there exists a preference relation R which rationalizes C. (R is then called a *rationalization* of C.) If a choice function C is rational with an acyclic rationalization, we say that C is *acyclic rational*. Similarly, if C is rational with an ordering rationalization, we say that C is *full rational*.

2.3. Let C be a choice function on (X, K) which is fixed once and for all. Two revealed preference relations R^* and R^{**} are induced from C as follows. We define a binary relation R^* on X by

$$R^* = \{(x, y): x \in C(S) \,\&\, y \in S \text{ for some } S \in K\} \tag{4}$$

and, when $(x, y) \in R^*$, we say that x is *revealed R^* preferred to y*. Similarly, we define

$$R^{**} = \{(x, y): x \in C(S) \,\&\, y \in S - C(S) \text{ for some } S \in K\} \tag{5}$$

and, when $(x, y) \in R^{**}$, we say that x is *revealed R^{**} preferred to y*. It is easy to see that R^* and R^{**} are related by the relations

$$P_{R^*} \subset P_{R^{**}} \subset R^{**} \subset R^*. \tag{6}$$

We have only to show that $P_{R^*} \subset P_{R^{**}}$, the remaining inclusions in (6) being obvious by definition. If $(x, y) \in P_{R^*}$, then we have

$$x \in C(S) \,\&\, y \in S \qquad \text{for some} \quad S \in K, \tag{7}$$

and

$$y \notin C(S') \qquad \text{or} \qquad x \notin S' \qquad \text{for all} \quad S' \in K. \tag{8}$$

If we apply (8) for $S' = S$, this (coupled with (7)) yields

$$x \in C(S) \,\&\, y \in S - C(S) \qquad \text{for some} \quad S \in K, \tag{9}$$

while (8) implies

$$y \notin C(S') \quad \text{or} \quad x \notin S' \quad \text{or} \quad x \in C(S') \qquad \text{for all} \quad S' \in K. \tag{10}$$

It follows from (9) and (10) that $(x, y) \in P_{R^{**}}$.

We now turn from our revealed preference relations to revealed preference axioms. A finite sequence $\{x^1, x^2, \ldots, x^n\}$ $(n \geqslant 2)$ in X is called an *H cycle of order n* if we have $(x^1, x^2) \in R^{**}$, $(x^\tau, x^{\tau+1}) \in R^*$ $(\tau = 2, \ldots, n-1)$ and $(x^n, x^1) \in R^*$. Similarly, a finite sequence $\{x^1, x^2, \ldots, x^n\}$ $(n \geqslant 2)$ in X is called an *SH cycle of order n* if we have $(x^1, x^2) \in R^*$, $(x^\tau, x^{\tau+1}) \in R^{**}$ $(\tau = 2, \ldots, n-1)$ and $(x^n, x^1) \in R^{**}$. In view of (6) it is clear that an *SH cycle of some order is an H cycle of the same order*. This being the case, the exclusion of an H cycle of any order excludes, a fortiori, the existence of an SH cycle of any order. We now introduce the following two revealed preference axioms.

HOUTHAKKER'S REVEALED PREFERENCE AXIOM. *There exists no H cycle of any order.*

STRONG AXIOM OF REVEALED PREFERENCE. *There exists no SH cycle of any order.*

Clearly, Houthakker's axiom is stronger than the strong axiom. We have shown in [10] that the strong axiom is necessary but not sufficient for full rationality and that it is sufficient but not necessary for acyclic rationality. In the final section we will argue that the above-stated Houthakker's axiom is a proper choice-functional counterpart of Houthakker's semitransitivity in demand theory.

2.4. We are now ready to put forward our theorem.

RATIONALIZABILITY THEOREM. *A choice function C is full rational if and only if it satisfies Houthakker's axiom of revealed preference.*

Proof of Necessity. If C is full rational with an ordering rationalization R, then we have (3). Suppose that there exists a sequence $\{x^1, x^2, \ldots, x^n\}$ $(n \geqq 2)$ such that $(x^1, x^2) \in R^{**}$ and $(x^\tau, x^{\tau+1}) \in R^*$ $(\tau = 1, 2, \ldots, n-1)$. Then there exists a sequence $\{S^1, S^2, \ldots, S^{n-1}\}$ in K such that $x^1 \in C(S^1)$, $x^2 \in S^1 - C(S^1)$, $x^\tau \in C(S^\tau)$, and $x^{\tau+1} \in S^\tau$ $(\tau = 1, 2, \ldots, n-1)$. Since C is full rational we then have $(x^1, x^2) \in P_R$ and $(x^\tau, x^{\tau+1}) \in R$ $(\tau = 2, \ldots, n-1)$, which entails $(x^1, x^n) \in P_R$, thanks to the transitivity of R. But this result excludes the possibility that $(x^n, x^1) \in R^*$, so that there exists no H cycle of any order.

Proof of Sufficiency. Let a *diagonal* Δ be defined by

$$\Delta = \{(x, x) : x \in X\}, \tag{11}$$

and define a binary relation Q by

$$Q = \Delta \cup T(R^*). \tag{12}$$

It is easy to see that Q is transitive and *reflexive*: $(x, x) \in Q$ for all x in X. Thanks to a corollary of Szpilrajn's theorem [3, Lemma 3] there exists an ordering R which subsumes Q; namely, there exists an ordering R such that

$$Q \subset R, \tag{13}$$

and

$$P_Q \subset P_R. \tag{14}$$

We are going to show that this R in fact satisfies

$$R^* \subset R, \tag{15}$$

and

$$P_{R^*} \subset P_R . \tag{16}$$

The former is obvious in view of $R^* \subset T(R^*)$, (12), and (13). To prove the latter we have only to show that $P_{R^*} \subset P_Q$, thanks to (14). Assume $(x, y) \in P_{R^*}$, which means $(x, y) \in R^*$ and $(y, x) \notin R^*$. From $(x, y) \in R^*$ it follows that $(x, y) \in Q$. It only remains to be shown that $(y, x) \notin Q$. Assume, therefore, that $(y, x) \in Q$. Clearly, $(y, x) \notin \Delta$, else we cannot have $(x, y) \in P_{R^*}$. It follows that $(y, x) \in T(R^*)$, which, in combination with $(x, y) \in P_{R^*} \subset R^{**}$, implies the existence of an H cycle of some order, a contradiction. Therefore (15) and (16) are valid.

Let an $S \in K$ be chosen and let $x \in C(S)$. Then $(x, y) \in R^*$ for all $y \in S$. In view of (15) we then have $x \in G(S, R)$. It follows that

$$C(S) \subset G(S, R). \tag{17}$$

Next let $x \in S - C(S)$ and take $y \in C(S)$, so that $(y, x) \in R^{**}$. If we have $(x, y) \in R^*$ it turns out that $\{x, y\}$ is an H cycle of order 2, a contradiction. Therefore we must have $(x, y) \notin R^*$, which, in view of $(y, x) \in R^{**} \subset R^*$, implies $(y, x) \in P_{R^*}$. Thanks to (16) we then have $(y, x) \in P_R$, entailing $x \in S - G(S, R)$. Therefore we obtain

$$G(S, R) \subset C(S). \tag{18}$$

As (17) and (18) are valid for any $S \in K$ we have shown that C is full rational with an ordering rationalization R. This completes the proof.

3. COMMENTS ON THE LITERATURE

It only remains to make some comments on the existing literature.

(i) Houthakker [4, pp. 162–163] introduced his simitransitivity axiom in terms of a demand function h on the family of competitive budgets. Let Ω, p, and m be the commodity space, a competitive price vector, and an income. Then h is a function on the family of all budget sets

$$B(p, m) = \{x \in \Omega : px \leq m\}. \tag{19}$$

We consider a sequence $\{x^\tau\}_{\tau=1}^{T}$ in Ω satisfying

$$x^\tau = h(p^\tau, p^\tau x^\tau) \qquad \text{for every} \quad \tau \in \{1, 2, \ldots, T\}, \tag{20}$$

$$x^{\tau+1} \in B(p^\tau, p^\tau x^\tau) \qquad \text{for every} \quad \tau \in \{1, 2, \ldots, T-1\}, \tag{21}$$

and

$$x^{\tau+1} \neq h(p^\tau, p^\tau x^\tau) \qquad \text{for } \textit{at least one} \quad \tau \in \{1, 2, \ldots, T-1\}. \tag{22}$$

Houthakker's semitransitivity then requires that $x^1 \notin B(p^T, p^T x^T)$. It will be noticed that what we called Houthakker's axiom of revealed preference in Section 2.3 is a natural reformulation of this requirement in terms of a choice function C.

(ii) What Samuelson [8, pp. 370–371] called the strong axiom is the same requirement as Houthakker's, save for the replacement of (22) by

$$x^{\tau+1} \neq h(p^\tau, p^\tau x^\tau) \qquad \text{for } \textit{every} \quad \tau \in \{1, 2, \ldots, T-1\}. \tag{23}$$

Our strong axiom of revealed preference in Section 2.3 is, it will be noticed, a natural extension of Samuelson's axiom in the context of a choice function.

(iii) We have shown in [10] that Hansson's strong axiom of revealed preference [3] is, despite its apparent difference, equivalent to that of ours in Section 2.3.

(iv) Richter [5, p. 637] argued that "the revealed preference notions employed in [the Weak Axiom of Samuelson and the Strong Axiom of Houthakker] are relevant only to the spacial case of competitive consumers, so that axioms also have meaning only in that limited context." However, these axioms can be and have been generalized beyond the narrow confinement of competitive consumers. Besides, Richter himself defined in [6] the weak and the strong axioms for a *single-valued* choice function. There is no reason, furthermore, that we should not consider these axioms in terms of a set-valued choice function.

In conclusion, it is hoped that our result will help to clarify the central role played by Houthakker's axiom in the whole spectrum of revealed preference theory.

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Distortion of Preferences, Income Distribution, and the Case for a Linear Income Tax*

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1. INTRODUCTION

The assumption of full information in economics and game theory often means that all utility functions (or preference orderings) are known to all. In many cases, like the analysis of competitive equilibrium, this information is not used by any participant and is thus irrelevant. However, in situations involving conflicts, externalities, or other public goods, the problem of revelation of preferences emerges in an essential way. The literature on this subject is small but growing rapidly (see [3–5, 7–13]). In the present paper I attempt to integrate the problem of revelation of preferences with the problem of income redistribution attacked by Aumann and Kurz [1, 2]. I will show that in the conflict situation proposed by Aumann and Kurz there is an incentive not to reveal the true utility function. Moreover, in the "distortion game" to be defined below, there exists a Nash equilibrium in which all individuals have a dominant strategy of pretending to have a linear utility function in income, thus revealing the same preference ordering. In the context of the Aumann–Kurz model this leads to the implication that redistribution is achieved via an income tax schedule which is the same for all agents and is linear in income.

2. THE AUMANN–KURZ MODEL AND THE PROBLEM OF REVELATION

In the Aumann–Kurz formulation there is a set T of agents, each with a utility function $u_t(y)$, where y is a real number indicating income. I take T to be a continuum; associated with T there is a nonatomic measure μ on the coalitions of T and endowment density $e(t)$, where $e(t)$ is bounded. It is also assumed that $\int_T e(t)\,\mu(dt) > 0$.

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If a coalition S is formed then it has total resources $\int_S e(t)\, \mu(dt)$, which it can distribute among its members. Thus if $y(t)$ is allocated to $t \in S$ then $u_t(y(t))$ is the utility of t, and feasibility requires $\int_S y(t)\, \mu(dt) = \int_S e(t)\, \mu(dt)$. However, due to the power of taxation, the democratic process provides that the majority is able to increase its wealth at the expense of the minority.

The essential nature of the Aumann–Kurz theory is that if S is a majority coalition (i.e., $\mu(S) > \frac{1}{2}$) then it can expropriate any part of the endowment of the minority coalition $T\backslash S$. However, the minority coalition has a powerful counterthreat: it can withdraw its endowment from the market either by destroying it (in the case of perishable goods) or by "striking" against the majority (in the case of labor services). In either case the counterthreat of the minority is to prevent the majority from taking possession of the endowment of the minority.

Now let U be an *admissible class* of utility functions which is defined by

(i) $u(x)$ is increasing, concave, continuously differentiable at positive values of the argument and continuous at 0;

(ii) $u(0) = 0$.

Now, adopting the Harsanyi–Shapley–Nash value as a solution concept, the main result of the Aumann–Kurz theory is that for any set of utility functions with $u_t(x) \in U$, $t \in T$, the income distribution game has a unique solution y which satisfies the equation

$$[u_t(y(t))/u_t'(y(t))] + y(t) = c + e(t), \tag{1}$$

where

$$c = \int_T [u_t(y(t))/u_t'(y(t))]\, \mu(dt). \tag{1a}$$

The critical fact, which gives rise to the current paper, is that in order for the allocation $y(t)$ to satisfy (1) it must be known that $u_t(\cdot)$ is the true utility function of agent t. Thus, suppose the agent announces that his utility function is $v_t(\cdot)$ rather than $u_t(\cdot)$. One possibility is that such an "announcement" is meaningless, since the true utility function will be revealed in his behavior in the bargaining process. If, however, the agent is able to pretend that his utility function is in fact $v_t(\cdot)$, and is able to *act accordingly*, then the Aumann–Kurz redistribution game may be played out with the utility function $v_t(\cdot)$ leading to the value allocation $x(t)$, which must satisfy

$$[v_t(x(t))/v_t'(x(t))] + x(t) = \hat{c} + e(t), \tag{2}$$

where

$$\hat{c} = \int_T [v_t(x(t))/v_t'(x(t))]\, \mu(dt). \tag{2a}$$

The above expectations of agent t are not unreasonable. In any cooperative game the agent expects a final compromise to be achieved, and for this reason he should consider any misrepresentation of his utility to be important only to the extent that it influences the final outcome. In this sense the agent can be expected to reveal a utility function $v_t(\cdot)$ and proceed to act upon this function until the final compromise is attained.

It is obvious that there are games in which the agent may be forced to face the outcome of his misrepresentation before a compromise is attained; in such games the expectations for an allocation such as the one defined by (2) would be unreasonable.

Returning to (2), I note that since there is a continuum of agents, $\hat{c}$ is a constant over which the agent has no control. Moreover, since the true utility function $u_t(y)$ is monotonic in y, it is clear that the agent will choose to announce a utility function $v_t(x)$ that would lead to the maximization of the outcome $y(t)$. It is then clear that the problem of revelation of preferences arises in the Aumann–Kurz theory due to the fact that the final outcome depends on both $u_t(y)$ and $e(t)$, while only $e(t)$ is observable and measurable. I am left with the conclusion that the agents have an incentive to distort their revealed utility functions.

3. THE DISTORTION GAME AND ITS EQUILIBRIUM

The distortion game is played by each agent revealing a utility function $v_t \in U$, where U is the common strategy space of all agents. Relative to the selection of v_t, $t \in T$, a corresponding pair $(c, x(t))$ is calculated leading to the payoff $u_t(x(t))$, $t \in T$. Now introduce the followng definition.

DEFINITION. A strategy $\{v_t^* \in U, t \in T\}$ is a Nash equilibrium in the distortion game if, for

$$c = \int_T [v_t^*(x(\tau))/v_t^{*\prime}(x(\tau))]\, \mu(d\tau),$$

$x(t)$ is the value allocation in the Aumann–Kurz income redistribution game and v_t^* is the optimal strategy for agent t.

(Note that I refer to $x(t)$ as *the* value allocation in the income redistribution game, since it is unique.)

THEOREM. *$v_t^*(x) = \gamma x$ with $\gamma > 0$ is the unique dominant strategy in U for all $t \in T$, resulting in a unique Nash equilibrium in the distortion game.*

Proof. First note that $v_t^* = \gamma x$, $\gamma > 0$ is an admissible utility function. Next I shall prove that $v_t^* = \gamma x$ is the solution of the following problem.

Select v_t^* that maximizes $x(t)$, where $x(t)$ is defined by

$$[v_t^*(x(t))/v_t^{*\prime}(x(t))] + x(t) = c + e(t).$$

To prove the optimality of $v_t^* = \gamma x$ assume that $g(x)$ is any admissible nonlinear function which is optimal. Let $\bar{x}_g(t)$ be the solution of the equation

$$g_t(\bar{x}_g(t)) + g_t'(\bar{x}_g(t))[\bar{x}_g(t) - c - e(t)] = 0. \tag{3}$$

Since $g_t(\bar{x}_g)) \geqslant 0$ and $g_t'(\bar{x}_g(t)) > 0$, it follows from (3) that

$$\bar{x}_g(t) - c - e(t) \leqslant 0. \tag{4}$$

If $\bar{x}_g(t) = 0$ then from (3), $c + e(t) = 0$; however, $c > 0$ (see [1, Eq. (4.7) and the condition $\int_T e > 0$] and $e(t) \geqslant 0$, which yields a contradiction. This proves $\bar{x}_g(t) > 0$. Moreover, if in (4) one had $\bar{x}_g(t) - c - e(t) = 0$ then (3) implies that $g_t(\bar{x}_g(t)) = 0$, contradicting our earlier conclusion. This proves that in (4) I actually have

$$\bar{x}_g(t) - c - e(t) < 0. \tag{4'}$$

I shall now show that there exists a linear function $f_t(x)$ such that if $\bar{x}_f$ is the solution of $f_t(x)$, i.e., if $\bar{x}_f$ satisfies

$$f_t(\bar{x}_f(t)) + f_t'(\bar{x}_f(t))(\bar{x}_f(t) - c - e(t)) = 0,$$

then

$$\bar{x}_f(t) > \bar{x}_g.$$

To prove the above, select

$$f_t(x) = [g_t(\bar{x}_g(t))/\bar{x}_g(t)]x, \tag{5}$$

which is well defined since $\bar{x}_g(t) > 0$. Now define

$$\begin{aligned} F_t(x) &= f_t(x) + f_t'(x)(x - c - e(t)) \\ &= \frac{g_t(\bar{x}_g(t))}{\bar{x}_g(t)}\, x + \frac{g_t(\bar{x}_g(t))}{\bar{x}_g(t)}\,(x - c - e(t)). \end{aligned} \tag{6}$$

It follows from the definition of $\bar{x}_f$ that

$$F_t(\bar{x}_f(t)) = 0. \tag{7}$$

However, I note that $F_t(x)$ is linear and monotonically rising in x. I shall next show that

$$F_t(\bar{x}_g(t)) < 0. \tag{8}$$

To see (7) note that

$$F_t(\bar{x}_g(t)) = g_t(\bar{x}_g(t)) + \frac{g_t(\bar{x}_g(t))}{\bar{x}_g(t)} (\bar{x}_g(t) - (c + e(t))). \tag{9}$$

Combining (3) with (9) I obtain that

$$F_t(\bar{x}_g(t)) = \left[\frac{g_t(\bar{x}_g(t))}{\bar{x}_g(t)} - g_t'(\bar{x}_g(t))\right] [\bar{x}_g(t) - c - e(t)]. \tag{10}$$

But by the strict concavity of $g_t(x)$ I have that

$$\left[\frac{g_t(\bar{x}_g(t))}{\bar{x}_g(t)} - g_t'(\bar{x}_g(t))\right] > 0,$$

while from (4′), $[\bar{x}_g(t) - c - e(t)] < 0$. This proves the inequality of (8). But now comparing (7) with (8) and noting that $F_t(x)$ is linear in x proves $\bar{x}_g(t) < \bar{x}_f(t)$, contradicting the optimality of $g_t(x)$. Thus, in the admissible class U the optimal solution is $v_t^*(x) = \gamma_t x$.

Now notice that if $f_t(x) = \gamma_t x$, then

$$f_t(x)/f_t'(x) = x,$$

and thus the optimal solution $x(t)$ must satisfy

$$x(t) = \tfrac{1}{2}(c + e(t)), \tag{11}$$

which is independent of γ_t. Thus $\gamma_t > 0$ is arbitrary and $f_t(x)$ may be taken as $f_t(x) = \gamma x$ all t.

Finally I claim that a linear utility function for all t is in fact a unique Nash equilibrium in the distortion game. Individual optimality has already been proved. Now note that if $v_t^* = \gamma x$ for all t (or $v_t^* = \gamma_t x$ $\gamma_t > 0$), then

$$c = \int [v_t^*(y(t))/v_t^{*\prime}(y(t))] \mu(dt) = \bar{e}.$$

Thus all agents will calculate the same c, and since $v_t^* = \gamma x$ is optimal with respect to any c, it follows that $v_t^* = \gamma x$ is in fact a dominant strategy in the distortion game. This concludes the proof of the theorem.

Note that in view of the fact that $c = \bar{e}$, (11) can be written as

$$\text{Tax of agent } t = e(t) - x(t) = \tfrac{1}{2}(e(t) - \bar{e}).$$

This characterization leads to the following conclusions:

(i) If $e(t) < \bar{e}$ then the tax of t is in fact a *subsidy* with a maximal value of $\frac{1}{2}\bar{e}$.

(ii) The marginal tax rate associated with this entire redistribution scheme is 50 %.

(iii) The subsidy–tax *schedule* is applicable to all agents regardless of their utility or wealth.

4. SOME QUALIFICATIONS AND INTERPRETATIONS

The theory proposed in this paper does raise some important questions which should be clarified. Note at the outset that whenever each of the agents reveals a utility function $v_t(x)$, the Aumann–Kurz income distribution game continues to be played *as if* $v_t(x)$ is the true utility function, since agent t is committed to act as if his utility function is truly $v_t(x)$. But then, one may ask, if everybody knows that false behavior was exhibited, why should the game be played in accordance with the Aumann–Kurz construction? Moreover, given that false behavior is exhibited, how could any solution concept be justified? Similarly, what should justify the restriction of exhibiting distorted preferences to any admissible class like U?

These are clearly fundamental questions applicable to all conflict situations in which distortion of preferences may occur. I do not believe that we can hope to find a general answer to such questions, since the answer must necessarily contain an essential grain of ethical considerations. This implies that any social organism must resolve such questions on the basis of "reasonable criteria." I am thus led to the belief that in any specific situation there may be specific considerations that would suggest what could be the resolution of these questions. In the case of the income redistribution game at hand, I can provide some justification for the procedure employed.

The income distribution game is inherently a one-dimensional game in which the conflict is over the single desired good. Given the monotonicity of preferences it is then clear that in our model all participants know the *ordinal* preferences of all the participants. What they do not know is the *intensities* of the preferences, and these are important elements in the specification of the outcome of the income distribution conflict. Note, however, that in distorting their preferences and "pretending to be what they are not," individuals take a great risk of inducing a most undesirable outcome. But, given that there is no way one can construct the intensity of their preferences from any observations, and given that individuals are willing to take the risk of falsification, it appears to me that the justification for the use of any solution concept is the same for the game with the true utility functions as for the game with the distorted preferences.

The question of restricting the admissible utility functions to the set U requires a different consideration. As can be seen from Eq. (1), the Aumann–Kurz theory suggests that in the determination of the income redistribution scheme, the critical dimension of the utility function is the ratio $u_t(x)/u_t'(x)$; Aumann and Kurz call this expression "the fear of ruin." To understand the reason for this terminology, consider a bet in which individual t can win \$1 but, on the other hand, can lose his entire fortune x. Let p be the *largest* probability of ruin (i.e., of losing x) which he would accept. Aumann and Kurz show that

$$p = u_t'(x)/u_t(x).$$

This means that a small p indicates a high fear of ruin and a large p indicates a small fear of ruin.

Returning now to the question of restricting the admissible set of utility functions to be U, I think that it is reasonable to expect that for the gamble discussed above the individual should exhibit *at least* risk neutrality, thus requiring p to provide at least a fair gamble. The most fearlessness the individual may possess can be indicated by a linear utility function. The set U does not admit a strictly convex utility function, since it leads to behavior which would be regarded, within the context of the present theory, as unreasonable and *unbelievable*.

The point made above gives an intuitive interpretation to my theorem: It says that *all participants pretend that they have no fear of ruin and all accept the compromise of "believing" each other and establishing a uniform income tax rate of* 50%, as shown in Eq. (11). Moreover, the claim that their utility functions are linear, is in fact a dominant strategy!

Finally, the reader may note two very appealing conclusions which emerge from the theory presented in this paper: First, the theory predicts that society will have a single, uniform income redistribution schedule applicable to all individuals. If the individual has no resources (i.e., $e(t) = 0$), he will receive a negative income tax of $\frac{1}{2}\bar{e}$: The subsidy or tax that the individual receives or pays is equal to

$$e(t) - x(t) = \tfrac{1}{2}(e(t) - \bar{e}).$$

Second, in view of the fact that individual members of coalitions may distort their true preferences in the social bargaining process, society acts *as if* all of us have equally intense appetites for money income and all of us are equally fearless of ruin!

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A Voting Model for the Allocation of Public Goods: Existence of an Equilibrium*

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It is well known that equilibria may not exist for majority voting over multidimensional policy spaces. This paper shows that certain institutional constraints can be imposed upon the voting process to ensure existence of a restricted equilibrium. A restricted equilibrium point must defeat only those points connected to it by one of an exogenously given set of linearly independent voting vectors. Using this procedure in a general equilibrium model to determine demands for public goods, existence of a general political equilibrium is proven. The equilibrium need not be Pareto optimal and may be manipulated by changing the vectors.

1. INTRODUCTION

Externalities in general and public goods in particular can cause the market mechanism to fail to allocate resources properly (in that its equilibria would not be Pareto optimal). Such market failure can be explained by the facts that prices reflect only private and not social costs and benefits and that individuals have incentives to reveal their preferences incorrectly, for example, by acting as free riders.

In these cases of market failure, economists have long believed that government intervention of some type may be an improvement over pure private markets. The government may either impose taxes and subsidies which equate private and social costs and hence lead to an efficient market outcome or it may shift some economic activities completely out of the market into the public sector. Practical implementation and full theoretical specification of either procedure would require the resolution of several difficulties, including

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that of securing a sufficient amount of correct information about preferences of individuals.

This paper develops a model of government activity of the second type, in which public good consumption levels are decided by government actions based upon democratic procedures, while private good consumptions are determined through the market. Despite much work, a fully specified model along these lines, with the completeness and sophistication of the general equilibrium analysis for a private good economy, has not been achieved in the literature. The difficulties are the familiar ones of securing truthful revelation of preferences, of finding a voting process for which an equilibrium exists, and of attaining Pareto optimality if an equilibrium does exist.

For purposes of this paper, the literature combining political procedures and public goods can be divided into two strands. The first is the integration of public goods into the general equilibrium framework, as typified by the work of Foley [13] and of Milleron [18]. This work concentrates on the economic structure and lacks a well-developed political mechanism. Foley, for example, introduces the concept of a public competitive equilibrium. Such an equilibrium must satisfy the usual conditions of profit maximization by firms, of consumer maximization over private goods, and of equality of demand and supply. In addition, the equilibrium public sector proposal, a vector of taxes and public goods, must be incapable of being overthrown. This means that, at equilibrium prices, no other tax and public goods vector can be chosen by the political mechanism. Foley leaves the mechanism generally unspecified, except to state that a necessary condition for a proposal not to be overthrown is that no other exist leading to higher utility for all individuals. In effect, then, his political mechanism is reduced to the unrealistic rule of unanimity. However, when specified in this manner the unanimity rule is not really determinate, since from any initial conditions any Pareto optimal point is a public competitive equilibrium. An alternative specification of the unanimity rule is made in the Lindahl model. An equilibrium here is a vector of tax rates and public goods such that given tax rates the public good bundle is preferred by all individuals to any other public good bundle. This formulation limits the set of equilibria for given initial conditions and, if truthful revelation of preference is assumed, leads to a Pareto optimal result. However, the incentives to misreveal preferences can be quite large, arising both from the use of the unanimity rule and from the fact that tax rates must be adjusted according to revealed public sector preferences. The failure to incorporate in these models more realistic political sectors is due to the nonexistence of equilibria of most voting processes. This nonexistence can be explained as a special case of Arrow's Impossibility theorem [1] applied to voting over multidimensional alternative spaces.

The second type of work considers the specific process of unrestricted simple majority voting, which allows any two feasible proposals to be opposed in voting and has all individuals voting in one body. Conditions are sought that ensure the existence of an equilibrium, a point unbeaten by any other in majority voting. A class of sufficient conditions ensuring consistent majority decision has been found, starting from the work of Black [5] on single peakedness. In work by Bowen [7], Buchanan [8], and Barr and Davis [3] dealing with economies with one public good, the resulting unidimensionality of the choice space along with normal assumptions about preferences leads to satisfaction of single peakedness, so that majority rule can be utilized consistently. Voting induces revelation of correct but insufficient information from individuals; hence, nonoptimality can result in these models.

For multidimensional spaces, the results of this type are extremely restrictive. Over these spaces, Kramer [5] shows that the sufficient conditions for majority consistency will almost never be satisfied. For a finite number of voters, Black and Newing [6], Plott [20], and Sloss [23] give sufficient conditions on the pattern of individual preferences at a point so that it is undefeated by any other. These conditions are unlikely to be met anywhere in the choice space. With an infinite number of voters, Tullock [25] and Davis, DeGroot, and Hinich [10] give other restrictions on preferences, but they are even less likely to be satisfied. Given the assumptions on preferences used in a standard general equilibrium model, an unrestricted majority rule equilibrium will not exist.

This paper, in the context of a general equilibrium model, will specify a type of restricted majority voting procedure which will be used to determine the demand for public goods. Other parts of the model, production and demand for private goods, are as in the standard economic models. The existence of an equilibrium for this model is proven and some comments are made about its properties.[1] In Section 2 the restricted voting procedure is described. Section 3 specifies the economic context in which this procedure is embedded. In Section 4, some properties of the model are summarized and some possible generalizations are mentioned. An outline of the proof of existence is given in Section 5, with the proof presented in Section 6.

2. THE VOTING MECHANISM

The policy space over which political decisions are made is a compact convex subset of R^m. Votes are taken only between points lying along certain directions. These directions are defined by a set of exogenously given linearly independent voting vectors $V = (v^1, v^2, \ldots, v^m)$, with m the dimension of the policy space. Two points z^1 and z^2 can be voted upon if and only if for one of the voting vectors there exists a λ such that $z^1 = z^2 + \lambda v^i$. For any given point z^0 in the policy space Z, the set of points which can be compared to it for voting purposes is the voting feasible set:

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[1] Work of a similar nature has also been done by Denzau and Parks [12].

DEFINITION I. The voting feasible set for z^0 is

$$v(z^0) = \{z: z \in Z \text{ and } z = z^0 + \lambda v^i, \text{ for some } \lambda \text{ and } v^i\}.$$

Votes are taken by strict majority rule, so that z^1 defeats z^2 if and only if the number of individuals who strictly prefer z^1 to z^2 is greater than the number who strictly prefer z^2 to z^1. This can be written

$$z^1Rz^2 \text{ iff } N(z^1P_iz^2) \geq N(z^2P_iz^1),$$

where R is weak social preference, P_i is strict preference of the ith individual, and $N(\ )$ is the number of individuals satisfying the condition in parentheses. A voting equilibrium is a point which defeats all points in its voting feasible set:

DEFINITION II. The voting equilibria for Z are

$$z^* = \{x: x \in Z \text{ and } N(xP_iz) \geq N(zP_ix), \forall z \in v(x)\}.$$

A simple two-dimensional three voter example is shown in the graph below (Fig. 1). The voting vectors, $V = (\binom{1}{0}, \binom{0}{1})$, are parallel to the coordinate axes. Each voter has a point of satiation labeled I, II, and III and preferences described by indifference contours, with those closer to the satiation point prefered. Point A is an equilibrium. Its voting feasible region is the two darkened lines, with A defeating by majority rule every other point on these lines.

This is the voting mechanism to be used to determine the demand for public goods. Some general comments on it will be made, since a more complete analysis is undertaken elsewhere.

First, the equilibrium notion used here was introduced by Black and

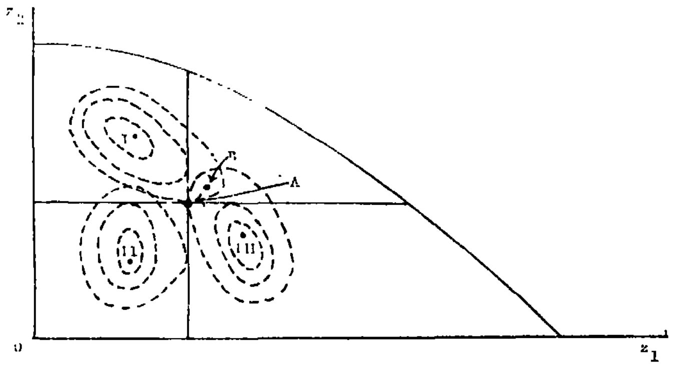

FIG. 1. A restricted voting equilibrium.

Newing [6] in the context of an abstract policy space with no prices or supply considerations. Kramer [16] then proved that such an equilibrium always exists in this situation. This equilibrium concept weakens the unrestricted majority equilibrium notion of Plott [20], which requires a point to be unbeaten by any other feasible one. Any Plott equilibrium will also satisfy this definition regardless of what set of voting vectors is chosen. This equilibrium will not, in general, be a Plott equilibrium, as can be seen in Fig. 1, where point *B* defeats *A* but is not in *A*'s voting feasible set.

Second, this mechanism can have an equilibrium because the voting feasible set is the union of unidimensional choice spaces. Under normal assumptions about the preference orderings of individuals, the property of single peakedness of the preferences of the set of individuals in society is attained along any line and, therefore, as shown by Black [5], there is a majority winner over it. What is sought here is a point which is simultaneously a winner in several directions.

Third, this model for connected choice spaces is in some ways an analog of the method of avoiding cyclical majorities over finite alternative spaces by fixing the order over which points are voted. Here, it is not the order of voting between lines which matters but the fact that the vectors impose an ordering on the points of the space. In the finite choice model, changing the order of votes between alternatives can change the winning point if an intransitivity exists. For this model, changing the set of voting vectors does in general change the equilibria. In fact, by suitable choice of vectors, almost any point in the policy space can be made an equilibrium.

Fourth, the voting vectors can be interpreted in various ways. In a general sense, they serve as a simple proxy for more complicated institutional structures found in actual voting situations. Unrestricted simple majority voting has the demand for public goods determined directly by vote of all individuals with no constraints on feasible votes. In reality, decisions are made by interacting branches of government and legislative bodies. Within any voting body, the results are strongly affected by committee structures, legislative leaders, and rules of procedure, formally or informally specified. These factors are extremely difficult to model in general. The imposition of voting vectors is at least a start in placing some institutional structure on the political process. Although highly simplistic, their use points out how an equilibrium can be created and manipulated through the institutions within which voting operates.

Specifically, the direction of the vectors can be considered to arise from packaging of commodities for strategic, administrative, or natural reasons. Since changing the vectors can alter the equilibrium, the ability to combine various commodities into a single voting issue may be one of the sources of power held by appropriations committees and legislative leaders. Administratively, large departments like HEW or HUD are involved in a wide variety of activities yet may have their appropriations voted upon in a single measure. Some public sector commodities by their nature will involve indivisibilities and externalities, and hence a decision to undertake a single project will provide various types of consumption services to individuals. For example, a vote on a dam is really a vote on some combination of energy, recreation, and irrigation. In cases of this type, if there are fewer vectors than commodities, then the voting process will still have equilibria. In fact, the set will be enlarged. However, if there are more vectors than commodities, then no equilibrium may exist.

Fifth, several of Arrow's conditions [1] are violated by this voting procedure. It determines only a winning or equilibrium point rather than a complete transitive social ordering, as required by Arrow. This mechanism is thus more comparable to a social decision function than to a social welfare function. However, a social preference relation can be derived from the voting process by intersecting the preferences defined by majority rule with the implicit ordering on the space imposed by the voting vectors. This social relation will be both incomplete and cyclic and therefore the condition of collective rationality will be violated. The incompleteness of the social preference relation also leads to violation of the Pareto principle. Points in the policy space unanimously preferred to the equilibrium can exist if they do not lie in its voting feasible set. In any feasible vote, unanimous choices are, of course, adopted. The other conditions of nondictatorship and independence of irrelevant alternatives are satisfied when, as assumed here, voting vectors are exogeneously specified. If the model is expanded so that the vectors are in some manner endogenously determined, then these may also be violated.

Sixth, in the finite choice model mentioned above, the possibility of strategic voting with individuals voting against their direct preference in a specific situation in order to move the final equilibrium to a more preferred point is well known. It will be assumed here that people vote their true preferences. This is more justifiable in the context of this paper than, for example, in the context of the Lindahl model. Fixed tax shares and majority rule limit the ability of an individual to gain from misrevelation of preference. False voting along a single vector, if the level of goods determined by votes along the other vectors is unchanged, can only hurt the individual. To attain an overall benefit from false voting, the individual would require much information about the preferences of others. Since, in most situations, sufficient information is unavailable, it is at least a good first approximation to neglect this possibility.

As a final point, the possibility of "trivial" equilibria arising from the restricted nature of voting must be noted. A point can be an equilibrium not because it defeats any others but because it is the unique element in its voting feasible set. An example of this is shown in Fig. 2.

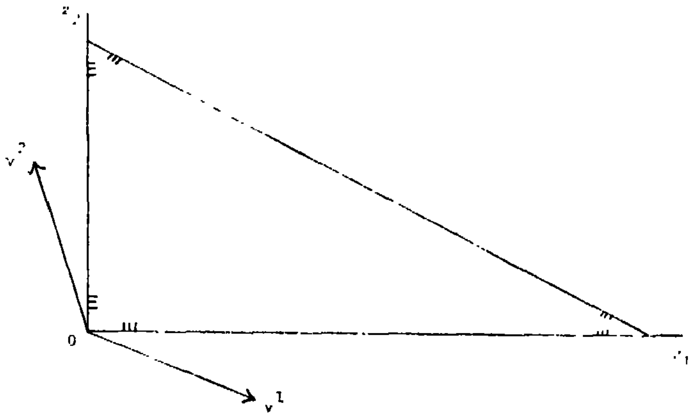

FIG. 2. A trivial equilibrium.

The origin is an equilibrium independent of preferences of individuals, since no other point can be offered against it. This type of situation can lead to difficulty in the proof of existence when the voting process is embedded into the economic framework. In the above example, if all public good prices are zero, with supply then assumed to be zero, the origin will remain a voting equilibrium and demand will equal supply. A general equilibrium would then just require that private prices clear private good markets.

Trivial equilibria can occur only at the boundary of the policy space, where no vectors point into the interior. If the voting vectors span the space, this cannot occur at a boundary point with a unique supporting hyperplane. At least one vector would point off the support plane and into the policy space. For trivial equilibria to exist, there must be kinks or corners in the boundary. For abstract political problems, these could simply be ruled out by assumption. In the economic context to be given, the linearities of the nonnegativity and budget constraints lead to corners. Restrictions on the voting vectors and on supply correspondences must be made to ensure that an equilibrium is nontrivial.

DEFINITION III. A voting equilibrium is nontrivial if

$$r(z^*) - \{z^*\} \neq \emptyset .$$

Definition III is weak in that it only requires the presence of real choice along one voting vector and not the more desirable assumption that choices exist along all vectors. This will at least be sufficient to rule out possibilities, as in Fig. 2 or in other cases where there is a strong desire by individuals for a change in public sector allocations. In special cases involving some strategic considerations in choice of vectors as mentioned above, this type of equilibrium may occur. In general, however, restricted choice equilibria are more likely to be a problem in proving existence in theoretical models than a difficulty in the real world.

3. The Economic Model.

The assumptions with the addition of the voting mechanism are basically those of a standard general equilibrium model with public goods, as in Foley [13].

The economy.

I. The economy has L individuals, where L is finite.

II. The economy has n private goods and m public goods. In consumption these are denoted by vectors $x \in \Omega^n$ and $z \in \Omega^m$ and in supply by $y \in R^{n+m}$, with the first n components private goods and the last m public. Public and private goods satisfy the following conditions on distribution among individuals:

$$\sum_{i=1}^{L} (x_j^i - \bar{x}_j^i) \leq y_j, \qquad j = 1, n,$$

$$z_j^i = z_j^k = y_{n+j}, \qquad \text{all } i, k; \quad j = 1, m.$$

III. There is a price vector $p = (p_x, p_z) \in \Omega^{n+m}$.

Producers face these prices directly, and consumers face them through a tax filter specified in assumption VII.

Production.

IV. There exists an aggregate supply correspondence $y(p) \in R^{n+m}$, positive for net outputs and negative for inputs, which is closed, bounded, convex, and upper semicontinuous. Public goods are never used as inputs in production, so that

$$\forall y \in y(p), \quad y_j \geqslant 0, \quad j > n.$$

Since this model differs from the standard model only on the demand side, the usual assumptions about production sets and profit maximizing firms are implicit, with supply correspondences assumed directly.[2]

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[2] The production assumptions are assumed to be as in Debreu [11].

Preferences.

V. The preferences of each individual i can be represented by a continuous quasi-concave utility function $U^i(x, z)$ which has no point of satiation.

Endowments.

VI. Each individual has a fixed initial endowment of private goods $\bar{x}^i \in \Omega^n$ with $\bar{x}^i > 0$ and a given continuous profit function $\psi^i(p)$ which is his share of total profits of the economy:

$$\sum_{i=1}^{L} \psi^i(p) = p\bar{y}, \qquad \bar{y} \in y(p).$$

Total income of individual i is

$$I^i(p) = p_x\bar{x}^i + \psi^i(p).$$

Government.

VII. Each individual is assigned a tax function $t_i(k_i)$ which states the percentage of the cost of the public sector paid by that individual. This function is continuous in k_i, where k_i is any set of parameters such as initial endowments or prices which within the scope of the model cannot be affected by actions of the individual.

VIII. The government budget must be balanced so that

$$\sum_{i=1}^{L} t_i = 1.$$

This tax treatment allows endowment taxes of any degree of progressivity or regressivity, poll taxes, a general sales tax, or any arbitrary lump sum tax system in which the percentage of the public sector budget paid by an individual is unaffected either by his decisions or by direct changes in the size or composition of the public sector.[3]

IX. Votes will be taken by strict majority rule only between public goods bundles which are

(a) feasible for all individuals;

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[3] Tax shares must not depend upon levels of public goods, since budget sets must be convex. Let the total tax function of the ith individual for the jth public good be $T_j^i(k_i, p, z_j)$. Then, for example, the set of (x, z_j) such that $p_x x + T_j^i(k_i, p, z_j) \leqslant I^i(p) - \sum_{l \neq j} T_l^i(k_i, p, z_l)$ is convex iff $T_j^i(k_i, p, z_j)$ is concave in z_j. Since $\sum_{i=1}^{L} T_j^i(k_i, p, z_j) = p_{n+j} z_j$ at all levels of z_j from the balanced budget property, each T_j^i must be linear in z_j or else for at least one individual i, T_j^i would be nonconcave. Total tax functions linear in z_j imply tax shares independent of z_j.

(b) related to each other by $z^2 = z^1 + \lambda v^i$ for some λ and some v^i belonging to an exogenously given set of linearly independent voting vectors $V = (v^1, \ldots, v^m)$.

Behavior.

X. In every separate decision, each individual acts as if he maximizes his utility function with respect only to those variables under consideration and subject to budget and voting constraints and to other variables and prices held fixed:

$$\operatorname*{Max}_{x, z_l} U^i(x, z), \qquad \text{s.t.} \quad p_x x + t_i(k_i)\, p_z z = p_x \bar{x}^i + \psi^i(p) = I^i(p),$$

where z_l is the subset of public goods variables under consideration.

This rules out strategic voting, since with other variables assumed constant, only direct preference between two situations may be considered. Strategic voting could succeed only through induced changes in the levels of the goods that are assumed constant. Note also that the individual is assumed to be a price taker in making decisions on both the public and private variables.

XI. Firms supply both public and private goods to the market, but:

(a) only individuals may purchase private goods;

(b) only the government may purchase public goods.

The social policy space is specified by assumptions VI to IX. For each individual, the feasible set of public goods is composed of those levels which, given his tax constant, do not exceed his budget constraint.

DEFINITION IV. The policy space of individual *i* is

$$Z_p^i = \{z : 0 \leq z \;\&\; t_i p_z z \leq I^i(p)\}.$$

The social policy space, to satisfy assumption IX(a), must be the set of points contained within every individual's policy space, and can easily be seen to be the intersection of the individual spaces.

DEFINITION V. The social policy space is

$$Z_p^s = \{z : 0 \leq z \;\&\; p_z z \leq \min_i (I^i(p)/t_i)\}.$$

As an example, consider the case of proportional income taxation. For each *i*, $t_i = I^i(p)/\sum_{j=1}^{L} I^j(p)$, so that $I^i(p)/t_i = \sum_{j=1}^{L} I^j(p)$, which is identical for all individuals. A proportional income tax gives rise to the largest social policy space of any type of taxation.

Voting feasible points $v(z^0(p))$ are points satisfying Definition I, with voting vectors given by IX(b) and the policy space Z_p^s.

Assumptions V, VI, and X specify how individuals choose their purchases of private goods in the market and how they vote for public goods through the political mechanism. Given a price vector and a level of public goods purchases, individuals then choose private goods to maximize their utility:

$$\operatorname*{Max}_{x} U^i(x, z), \qquad \text{s.t.} \quad p_y x \leqslant I^i(p) - t_i p_z z, \qquad x \geqslant 0.$$

Denote the maximizing values of x by $x^*(p, z)$. In deciding how to vote between two feasible public goods vectors, individuals compare the utility they would receive after picking their maximal private bundle for each public good vector. Preferences over points in the individual's policy space can be described by the derived utility function

$$H_p^i(z) = U^i(x^*(p, z), z).$$

$H_p^i(z)$ is defined only over points in the policy space and varies with prices.

For each price vector, a political equilibrium $z^*(p)$ is a vector of public goods satisfying Definition II for policy space Z_p^s, voting feasible points $v(z^*(p))$, and preferences $H_p^i(z)$, $i = 1, L$. This is in essence a point on the demand curve for public goods.

A general political equilibrium in which the public good level is a voting equilibrium and all markets, public and private, clear, is defined as follows.

Definition VI. A nontrivial general political equilibrium is a price system and public goods vector $(\tilde{p}, \tilde{z})$ with which are associated vectors $\tilde{x}^i$, $i = 1, L$, and $\tilde{y}$ such that:

(a) $\tilde{z}$ is a nontrivial voting equilibrium. That is

(i) $\tilde{z} \in Z_{\tilde{p}}^s$ so that $\tilde{z} \geqslant 0$ and $\tilde{p}_z \tilde{z} \leqslant \min(I^i(\tilde{p})/\tilde{t}_i)$,

(ii) $N(\tilde{z} P_i z) \geqslant N(z P_i \tilde{z})$ for all $z \in v(\tilde{z})$, that is for all $z \in Z_{\tilde{p}}^s$ and $z = \tilde{z} + \lambda v^i$ for some λ and v^i;

(iii) $v(\tilde{z}) - \{\tilde{z}\} \neq \phi$ so that for some v^i there exists a $\bar{\lambda} \neq 0$ with $z = \tilde{z} + \bar{\lambda} v^i \in Z_{\tilde{p}}^s$.

(b) All markets, public and private, clear. That is, $\tilde{y} \in y(\tilde{p})$ and

(i) $\tilde{z}_k = \tilde{y}_{n+k}$, $k = 1, m$;

(ii) $\sum_{j=1}^{L} \tilde{x}_i^j \leqslant \tilde{y}_i + \sum_{j=1}^{L} \bar{x}_i^j$, $i = 1, n$, with equality if $\tilde{p}_i > 0$. For all individuals j, $U^j(\tilde{x}^j, \tilde{z}) \geqslant U^j(x, \tilde{z})$ for all $x \geqslant 0$ and $\tilde{p}_y x \leqslant I^j(\tilde{p}) - \tilde{t}_j \tilde{p}_z \tilde{z}$.

Assumptions I to XI are at the level of generality found in the standard general equilibrium model. Those assumptions relating to private good consumption and production are sufficient for the existence of a private competitive equilibrium. Those relating to the voting process are sufficient for the existence of a separate voting equilibrium. Other assumptions are specified to tie these two models together in an interesting fashion. It would be desirable to prove the existence of the combined general political equilibrium under them. However, with just these assumptions some technical problems arise in the proof due to discontinuities in the demand functions at special points and due to the possible occurrence of trivial equilibria. In order to prove existence, some additional restrictions on some of the variables or parameters are needed. These are detailed below in technical assumptions TI to TIV, which were chosen more for their simplicity in avoiding potential difficulties than for their generality, although they do not appear to be overly restrictive.

TI. $U^j(x, z)$ is strictly quasi-concave.

This modifies assumption V and is stronger than needed to prove the existence of a private good competitive equilibrium, but is necessary to prove the existence of a voting equilibrium in an arbitrary policy space.

TII. (a) $v_j^i \geq 0$, all i, j.

(b) There exist an i and k with $v_k^i > 0$ and $v_j^i = 0$, all $j \neq k$.

The voting vectors given in assumption IX(b) are now required to satisfy more than linear independence. TII(a) requires that all vectors point into the nonnegative orthant so that along a vector any proposed increases in the levels of some public goods must be funded by tax increases alone and not by a direct decrease in the levels of any others. Indirectly, substitution effects, operating through both preferences and budget constraints of individuals, may well lead to reductions in other commodity levels. This assumption seems a reasonable approximation to practices in most actual legislative situations. Only rarely are bills proposed directly linking a rise in one department's appropriations to a fall in those of some other. TII(b) says that at least one of the voting vectors must coincide with one of the coordinate axes so that some voting direction affects the level of a single public good only.

TIII. There exists a correspondence $S: p \rightarrow 2^{R^m}$ which satisfies:

(i) $S(p) \subseteq Z_p^s$.

(ii) For each p, $S(p)$ is closed, convex, and homogeneous of degree zero in prices.

(iii) For all j, if $z \in S(p)$ and $I^j(p) > 0$, then $I^j(p) - t_j p_z z > 0$.

(iv) $S(p)$ is continuous in p.

(v) For all i, $Z^i(p, z) = \{\hat{z}: \hat{z} \in S(p) \text{ and } \hat{z} = z + \lambda v^i, \text{ for some } \lambda\}$ is continuous in p and z, for $z \in S(p)$

A public goods vector z is feasible only if $z \in S(p)$.

Feasibility is restricted beyond the requirements of assumption IX(a), which specified only that public goods must lie within the budget sets of all individuals. TIII removes some of these points from the feasible set as a cost of achieving the continuity part of the assumption. As will be seen in the proof, the social policy space, while closed, convex, and homogeneous of degree zero, violates the continuity requirement at certain boundary points when one of the voting vectors becomes parallel to the budget plane. Below, a specific construction of a correspondence satisfying TIII is given for which the set of points in the range of the correspondence but not in $Z_p{}^s$ is empty for most p and can be made arbitrarily small when nonempty. The impact of the assumption in so far as it truncates the social policy space can be made negligible. Further, for this, or for some other construction it may be possible to add assumptions on preferences or production which ensure that the equilibrium is strictly interior to $S(p)$, so that the truncation is not binding in equilibrium.

The construction is as follows. Let $\{B\}$ be the collection of sets of voting vectors with the property that $\alpha \in \{B\}$ iff there exists a component i with $v_i{}^k = 0$ for all vectors $k \in \alpha$. For any $\alpha \in \{B\}$ define the surrogate price vector s^α by $s_i{}^\alpha = 0$ if $v_i{}^k > 0$, some $k \in \alpha$, and $s_i{}^\alpha = p_{n+i}$, if $v_i{}^k = 0$, all $k \in \alpha$. Then $s^\alpha \cdot v^k = 0$, all $k \in \alpha$. The level of income constraining feasibility is also adjusted. For any vector v^k let the set of components i with $v_i{}^k > 0$ be denoted $\{i/k\}$. For any small $\epsilon > 0$, let $\epsilon^k(p) = \min(\epsilon \cdot (p_{n+i}/\sum_{j=1}^{n} p_{n+j}))$, for all $i \in \{i/k\})$ and let $\epsilon^\alpha(p) = \sum_{k \in \alpha} \epsilon^k(p)$. Finally, let $\bar{I}(p) = (1 - \delta) \min_j [I^j(p)/t_j]$. Then

$$S(p) = \{z_i : z \geq 0,\ p_z z \leq \bar{I}(p),\ \&\ s^\alpha z \leq [1 - \epsilon^\alpha(p)]\, \bar{I}(p),\ \text{all } \alpha \in \{B\}\}$$

satisfies all conditions of TIII, as shown in Slutsky [24].[4]

TIV. $y_{n+k} > 0$ iff $p_{n+k} > 0$, $\quad y \in y(p)$.

The general restrictions on supply correspondences given by assumption IV are strengthened. TIV means that if the price of a public good is positive, at least some of it will be produced and if the price is zero, none will be. This type of behavior could arise from underlying production functions satisfying Inada-type conditions utilized in growth theory [14].

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[4] Copies of this proof are available from the author.

4. PROPERTIES AND EXTENSIONS

In Section 2 the properties of the voting process over an abstract policy space were reviewed. In this section, also in a general way, some properties of the general political equilibrium will be discussed. Then some possible extensions of the model will be considered.

As might be expected, a general political equilibrium need not be Pareto optimal. Nonoptimality arises from the failure of the voting process to utilize sufficient information. This occurs in two different ways. First, as mentioned in Section 2, social preferences are incomplete, with votes taken only between points connected by one of the voting vectors. No information about points not in each other's voting feasible set is elicited. This makes possible the existence of points unanimously preferred to the equilibrium, as long as they are not in its feasible set. Second, the proper allocation between public and private goods may fail to be achieved. The Samuelson [21] necessary conditions for optimality, that the sum of the marginal rates of substitution between a public and a private good must equal the marginal rate of transformation, can fail to be satisfied. As known from the work of Black [5], along each vector the majority winner can be found by taking the median of the constrained best points of each individual. Information about the marginal rate of substitution of each individual at the median is not elicited. Thus, it may be possible to change the level of public goods from the equilibrium level, tax those whose utility is raised, and fully compensate those made worse off, hence raising the utility of all individuals. This is a well-known result in models with a single public good.

These types of nonoptimality are not easily remedied. They arise for the very reasons that make the process specified in this paper workable. An equilibrium exists only because social preferences are incomplete. Tax shares are fixed exogenously in a way unrelated to marginal rates of substitution. Adjusting tax rates, so that for each individual they equal his marginal rate of substitution, could ensure satisfaction of the Samuelson optimality conditions. It would also create large incentives for individuals to misreveal their preferences.

Proposals to circumvent these difficulties have been made. For example, Buchanan and Tullock [9] suggest that allowing logrolling removes the inefficiencies of separate issue voting. Under logrolling, in either explicit or implicit form, individuals trade votes on the different issues, giving up votes on issues unimportant to them to gain votes on important ones. Such a process has not been specified rigorously in a manner consistent with the model given here.[5] Various problems with this approach can be seen. In

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[5] In Wilson [26], for example, preferences over the policy space are assumed to be independent. This can be shown to be extremely restrictive, if not inconsistent with the general equilibrium framework used here (see [24]).

Fig. 1, individual I would be willing to trade his vote along vector v^1 for individual III's vote along v^2 to bring about a better result for both than point *A*. In effect, they form a coalition to reach point *B*. But, since no unrestricted majority equilibrium exists, another coalition would exist to trade votes and defeat *B*. Logrolling breaks down the separation among issues and removes the basis upon which existence was proven. Even if, under special circumstances, equilibrium did exist, it might not be an improvement on efficiency grounds. Any benefits from increased efficiency of the voting equilibrium might be lost by increased distortions between public and private goods. To achieve overall optimality, vote buying as well as vote trading would be required. Objections to such a procedure could be made on grounds of feasibility and of equity.

The possibility of nonoptimal equilibria seems to be an ineradicable part of this model. The consequences of this are mitigated by several considerations. The inefficiencies are only potential. They are not forced in the manner of a distortionary tax. No apparent bias seems to arise for the equilibrium either to undersupply or oversupply public goods as compared to the set of optimal allocations. Further, given the difficulties inherent in allocation of public goods, the proper comparison should not be with unobtainable optimal points, but with what would be achieved by the private market unaffected by government intervention. For example, the equilibria in a model with dependent consumer preferences in which each individual assumes that the levels of public goods purchased by other individuals are unaffected by his own purchases, as discussed in McKenzie [17] or Arrow and Hahn [2], could be compared to equilibria of this model. Clearly at the level of abstraction of these models general statements would be difficult, if not impossible, to make, but results might be attained under simplifying assumptions. It may be that any inefficiencies of this model are an acceptable cost of overcoming intransitivities and nonexistence of equilibria.

To extend the model by allowing for strategic factors, impure public goods, and public production goods would be desirable. Each has problems, some of which it may not be possible to overcome. Strategic voting can easily lead to nonexistence of equilibrium. With impure public goods allowed, the possibility of simultaneous purchase by individuals must be considered. To vote on production goods, individuals would need much information on profit functions. Also, in voting rationally they would consider the effects on prices and wages of changes in public input levels and thus would violate price taking. The tax structure could also be modified in several ways. Taxes could be made different on different goods, thus allowing for local public goods and different voting districts. In various ways, the tax structure would be made endogeneous by having votes over parameters upon which the rates depend. Care here would be needed so that equilibria would still exist.

5. OUTLINE OF PROOF

The proof is a modification of standard existence proofs for general equilibrium models, especially those for economies with public goods or externalities. As an initial simplification, it is assumed that the voting vectors are parallel to the coordinate axes, so that for each i there exists a j with $r_j^i \neq 0$ and $r_k^i = 0$, all $k \neq j$. Bounded consumption sets for public and private goods are defined to bound demand functions. An individual's choices are feasible iff they lie in the consumption set within his budget constraint and if public goods lie in the set $S(p)$, the truncated social policy space defined in TIII.

Demand functions for each individual for private goods which depend upon prices and the levels of public goods, are found by maximizing $U^i(x, z)$ over feasible x with z fixed. Pseudo-demand functions, so called because they are never directly expressed in the market, for each public good z_j, which depend upon prices and the levels of the other public goods $z(j)$, are found by maximizing over feasible x and z_j for fixed $z(j)$. Both types of demand functions are shown to satisfy standard properties. Continuity requires assumption TIII since it is shown that with $z(j)$ fixed, the set of feasible z_j would otherwise have a discontinuity at price–quantity situations where $p_{z(j)}\, z(j) = \min(I^i(p)/t^i)$ and $p_{n+j} = 0$. Such a discontinuity is similar to the one in a private good economy that occurs when the price of a commodity and the value of initial endowments simultaneously go to zero. In the private good economy, various direct assumptions can be made to rule out such a possibility (for example, by making the endowment of each individual of every commodity strictly positive). Here, to remove the discontinuity requires more complicated constructions of the type discussed in TIII.

From individual demands, aggregate and social demands are derived. Aggregate private demand is the sum of individual demands as functions of prices and public goods. Social demand correspondences for public goods are found by taking the median set over individual pseudo-demand functions. This median set is shown to be compact, convex, and uppersemicontinuous in prices and other good levels.

A mapping is then constructed from a quantity–price space into itself. The price space, as usual, is the unit simplex, and the quantity space is the cross product of the consumption and production sets for private and public goods. The mapping into prices is the standard excess-demand mapping. The mapping into the production set is the supply correspondence given in assumption IV. The aggregate and social demand correspondences defined above map prices and public good quantities into the private and public consumption sets. These latter mappings for any set of prices are defined only for public good bundles in $S(p)$. They must be extended so as to be defined over all points in the public goods consumption set. The mapping thus constructed is shown to satisfy the conditions of the Kakutani fixed point theorem and has a fixed point.

To complete the proof, the fixed point is shown to be a nontrivial general political equilibrium. First, the public goods vector is shown to lie in the set $S(p)$ and to be a majority rule winner along all vectors. It is nontrivial, since at least one vector can be shown to point into the policy space. Then the fixed point supply vector is shown equal to the fixed point demand vector in all components, public and private.

Finally, it is indicated how the proof can be modified to handle voting vectors not parallel to the coordinate axes. This can be done by transforming variables from z into $\bar{z}$ with $z = V\bar{z}$, where V is the matrix of voting vectors. The utility function and the consumption and budget sets must be transformed in accordance with this. Desirable properties, such as continuity and quasi-concavity of utility functions and compactness and convexity of consumption sets, are maintained. It should be noted that the transformed consumption set in $\bar{z}$ space will not coincide with the nonnegative orthant unless all voting vectors are parallel to the coordinate axes in the original public goods space. Maximizing transformed utility over one transformed variable with the level of others fixed is the same as picking a best point along one of the voting vectors given some initial quantity point as origin. Demand functions can be found as above.

6. Proof of Existence[6]

The consumption sets for private and public goods are the nonnegative orthants bounded from above by some large constant in order to ensure bounded demand functions. These consumption sets are:

$$E_x = \{x\colon 0 \leqslant x_i \leqslant k\},$$
$$E_z = \{z\colon 0 \leqslant z_i \leqslant k\}.$$

The constant k can be made arbitrarily large so as to be nonbinding in equilibrium.

Private good demand of the jth individual can be found by the following maximization problem:

$$\max_{x^j} U^j(x^j, z) \qquad \text{s.t.} \quad p_x x^j \leqslant I^j(p) - t_j p_z z, \qquad x^j \in E_x .$$

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[6] The following notational conventions are used. Small letters denote vectors and functions; capitals denote sets, matrices, and some functions. For vector $x = (x_i, x(i)) \in R^n$, x_i is the ith component and $x(i) \in R^{n-1}$ is for the vector with the ith component removed. Superscripts distinguish different vectors. $\text{proj}_i(Y)$ and $\text{proj}_{(i)}(Y)$ are the projections of set Y on the ith axis and on all axes but the ith.

Denote the demand functions determined in this problem as $x^j = h^j(p, z)$, $x^j \in R^n$. This problem is feasible for z such that $I^j(p) - t_j p_z z > 0$, but to be consistent among individuals only $z \in E_z \cap Z_p^s$ will be considered. Actually the set $z \in E_z \cap S(p)$ must be used to avoid a discontinuity which can arise when a private good price and effective income of some individual, $I^j(p) - t_j p_z z$, both equal zero, just as can happen in the standard pure private good economy. Note that this can occur on the full set even if $I^j(p) > 0$, all j, since it is also necessary for every individual to have positive income left after tax payments to spend on private goods.

LEMMA I. For $z \in E_z \cap S(p)$, $h^j(p, z)$, for all j, are (i) nonnull, (ii) single valued, (iii) bounded, (iv) homogeneous of degree zero in prices, (v) continuous where $I^j(p) > 0$, all j.

Proof.[7] (i) For $z \in E_z \cap S(p)$, $p_z z < \min(I^j(p)/t_j)$, implying $t_j p_z z < I^j(p)$, all j; thus z is feasible. The continuous function $U(x, z)$ attains a maximum over nonempty compact constraint sets.

(ii) The constraint set is convex and $U(x, z)$ is strictly quasi-concave. Therefore the maximizing values must be unique.

(iii) Set E_x bounds the variables.

(iv) All constraint functions, $S(p)$ and the budget constraint, are homogeneous of degree zero in prices.

(v) Since (ii) holds, consider sequences of (p^n, z^n, x^n) for which (1) $p^n \to p^0$ and $z^n \to z^0$, $z^n \in E_z \cap S(p^n)$ and (2) $x^n = h(p^n, z^n)$ and $x^n \to x^*$. Continuity follows if $x^* = h(p^0, z^0)$. To satisfy this, x^* must be both feasible and maximal at (p^0, z^0). Note that since E_z is compact and the graph $(p, S(p))$ is closed, $z^0 \in E_z \cap S(p^0)$ with feasible x existing for all individuals at (p^0, z^0). Since $I(p)$ and t are continuous in prices, if x^* is not feasible, so that $p_x^0 x^* > I(p^0) - t^0 p_z^0 z^0$, then after some n in the sequence, $p_x^n x^n > I(p^n) - t^n p_z^n z^n$ contradicting $x^n = h(p^n, z^n)$. Hence x^* is feasible.

Now consider any x' feasible for (p^0, z^0) so that $p_x^0 x' \leqslant I(p^0) - t^0 p_z^0 z^0$. For $\lambda^n = \min[1, (I(p^n) - t^n p_z^n z^n)/p_x^n x']$, $p_x^n \lambda^n x' \leqslant I(p^n) - t^n p_z^n z^n$. Then $x'^n = \lambda^n x'$ is feasible with $x'^n \to x'$, since $(I(p^0) - t^0 p_z^0 z^0)/p_x^0 x' \geqslant 1$. Since x^n is maximal with $U(x^n, z^n) \geqslant U(x'^n, z^n)$, for all n, continuity of U implies that $U(x^*, z^0) \geqslant U(x', z^0)$ is true for any feasible x'. Hence x^* is maximal.

Q.E.D.

Next, pseudo-demand functions for public goods will be defined. The demand for public good i by individual j is found by the maximization problem

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[7] In the proofs of Lemmas I and II, the superscript j on demand functions $h^j(p, z)$, $h^{ij}[p, z(i)]$, and $f_i^j[p, z(i)]$ referring to individuals will be dropped for notational simplicity. These lemmas deal with the behavior of single individuals so that no confusion should result.

$$\max_{x,z_i} U^j(x, z_i, z(i)) \quad \text{s.t.} \quad p_x x + t_j p_{n+i} z_i \leq I^j(p) - t_j p_{z(i)} z(i),$$
$$x \in E_x,$$
$$(z_i, z(i)) \in E_z \cap S(p).$$

Denote the pseudo-demand functions determined in this problem as $z_i^j = f_i^j(p, z(i))$, and $x^{ij} = h^{ij}(p, z(i))$. It is in order to ensure continuity of these functions that the correspondence $S(p)$, truncating the social policy space, is needed. If defined over the untruncated policy space, a discontinuity in h^{ij} similar to the one in h^j discussed above can occur. In addition, there can exist discontinuities in f_i^j in situations with $p_z z = \min(I^j(p)/t_j)$ and $p_{n+i} = 0$. This happens because the budget plane becomes parallel to a voting feasible direction when $p_{n+i} = 0$. The feasible set of z_i at $p_{z(i)} z(i) = \min(I^j(p)/t_j)$ jumps from the single point $z_i = 0$ when $p_{n+i} > 0$ to the entire line $0 \leq z_i \leq k$ when $p_{n+i} = 0$. This can be seen in the example in Fig. 3.

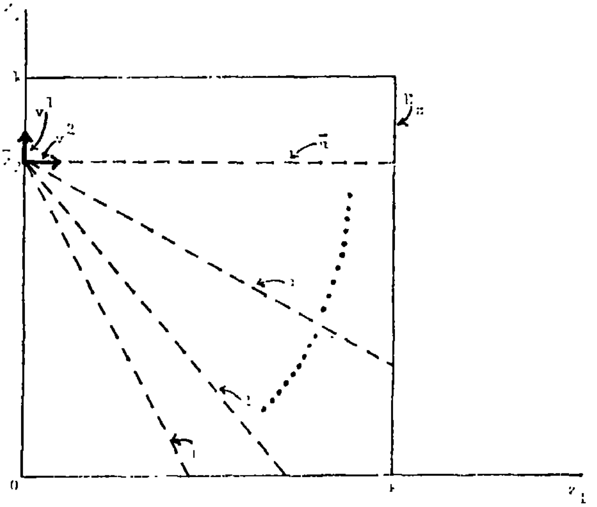

FIG. 3. A boundary discontinuity.

Lines 1, 2,..., n are a sequence of boundaries of the social policy space rotating around the point $(0, \bar{z}_2)$ converging to line $\bar{n}$ as $I^j(p)$ and p_{n+2} remain constant while p_{n+1} goes to zero. When $p_{n+1} > 0$, only $z_1 = 0$ is feasible at $z_2 = \bar{z}_2$. When $p_{n+1} = 0$, any z_1 in E_z is feasible. At all points other than these, the demand functions are continuous. Since continuity is required at all points, the truncation set $S(p)$ must be utilized.

LEMMA II. For $z(i) \in \text{proj}_{(i)}(E_z \cap S(p))$, $f_i^j(p, z(i))$, for all i and j, are (i) nonnull, (ii) single valued, (iii) bounded, (iv) homogeneous of degree zero in prices, (v) continuous where $I^j(p) > 0$, all j.

Proof. (i)–(iv) follow as in Lemma I.

(v) Consider sequences of $(p^n, z^n(i), z_i^n, x^n)$ for which

(1) $p^n \to p^0$, $z^n(i) \to z^0(i)$, $z^n(i) \in \text{proj}_{(i)}(E_z \cap S(p^n))$,

(2) $z_i^n = f_i(p^n, z^n(i))$, and $z_i^n \to z_i^*$,

(3) $x^n = h^i(p^n, z^n(i))$ and $x^n \to x^*$.

Continuity follows if $z_i^* = f_i(p^0, z^0(i))$ and $x^* = h^i(p^0, z^0(i))$. This requires showing that z_i^* and x^* are feasible and maximal for p^0 and $z^0(i)$. Feasibility follows just as in Lemma I.

To show that x^* and z_i^* are maximal, let z_i' and x' be any feasible points for p^0 and $z^0(i)$, that is for which $(z_i', z^0(i)) \in E_z \cap S(p^0)$ and $p_x^0 x' + t^0 p^0_{n+i} z_i' \leq I(p^0) - t^0 p^0_{z(i)} z^0(i)$. z_i' may not be feasible for earlier points $(p^n, z^n(i))$ in the sequence. However, by assumption TIII, the set of points $\{z_i : (z_i, z(i)) \in S(p)\}$ is continuous, and thus there must exist a sequence of points $z_i'^n$ with $(z_i'^n, z^n(i)) \in E_z \cap S(p^n)$ and $z_i'^n \to z_i'$. For sequences $(z_i'^n, z^n(i))$, just as in Lemma I, there must exist a sequence of $x'^n \to x'$ with $p_x^n x'^n + t^n p^n_{n+i} z'^n \leq I(p^n) - t^n p^n_{z(i)} z^n(i)$. Thus $(x'^n, z_i'^n)$ is feasible all n with $U(x^n, z_i^n, z^n(i)) \geq U(x'^n, z_i'^n, z^n(i))$. By continuity of U, $U(x^*, z_i^*, z^0(i)) \geq U(x', z_i', z^0(i))$, making x^* and z_i^* maximal. Q.E.D.

There are L functions, $z_i^j = f_i^j(p, z(i))$. Let the social demand function $z_i^s = f_i^s(p, z(i))$ be defined as the median set:

$$f_i^s(p, z(i)) = \{z_i : N(f_i^j(p, z(i)) \geq z_i) \geq L/2 \ \&\ N(f_i^j(p, z(i)) \leq z_i) \geq L/2\}.$$

LEMMA III. *For* $z(i) \in \text{proj}_{(i)}(E_z \cap S(p))$, $f_i^s(p, z(i))$ *is* (i) *nonnull,* (ii) *compact,* (iii) *convex,* (iv) *homogeneous of degree zero in prices and* (v) *upper semi continuous where* $I^j(p) > 0$, *all* j.

Proof. (i) At each $(p, z(i))$, order the f_i^j by size and relabel so that $j < k$ implies $f_i^j \geq f_i^k$. For the integer k such that $L \leq 2k \leq L + 1$, $f_i^k \in f_i^s$.

(ii) At each $(p, z(i))$, the set f_i^s must be bounded, since all f_i^j lie in a bounded set. Let there exist a sequence of $z_i^n \in f_i^s$ with $z_i^n \to z_i'$. With the relabeling of individuals, $z_i^n \geq f_i^j$, all $j \leq k$, and $z_i^n \leq f_i^j$, all $j \geq k$, for all n. Since $z_i' \geq f_i^j$, all $j \leq k$, and $z_i' \leq f_i^j$, all $j \geq k$, $z_i' \in f_i^s$ and f_i^s is closed and hence compact.

(iii) At any $(p, z(i))$, consider z_i', $z_i'' \in f_i^s$ with $z_i' > z_i''$. For any $0 \leq \lambda \leq 1$, let $z_i^\lambda = \lambda z_i' + (1 - \lambda) z_i''$, so that $z_i' \geq z_i^\lambda \geq z_i''$. Since $N(z_i'' \geq f_i^j) \geq L/2$ implies $N(z_i^\lambda \geq f_i^j) \geq L/2$, and since $N(z_i' \leq f_i^j) \geq L/2$ implies $N(z_i^\lambda \leq f_i^j) \geq L/2$, then $z_i^\lambda \in f_i^s$.

(iv) Since each $f_i^j(p, z(i))$ is homogeneous of degree zero in prices, the median set is also.

(v) To show upper semicontinuity a new correspondence, $M(p, z(i))$, will be defined for which upper semicontinuity follows directly from a theorem of Berge [4]. $M(p, z(i))$ will then be shown equivalent to $f_i^s(p, z(i))$. First, for each individual j, define $A_i^j(p, z(i))$ and $B_i^j(p, z(i))$ as

$$A_i^j(p, z(i)) = \{z_i : (z_i, z(i)) \in E_z \cap S(p) \quad \text{and} \quad z_i \geqslant f_i^j(p, z(i))\},$$

$$B_i^j(p, z(i)) = \{z_i : (z_i, z(i)) \in E_z \cap S(p) \quad \text{and} \quad z_i \leqslant f_i^j(p, z(i))\}.$$

Consider a sequence of $p^n \to p^0$, $z^n(i) \to z^0(i)$, and $z_i^n \to z_i^0$ with $z_i^n \in A_i^j(p^n, z^n(i))$. Then $z_i^0 \geqslant f_i^j(p^0, z^0(i))$, since all f_i^j are continuous in p and $z(i)$. Since $S(p)$ is a continuous correspondence, the graph $(p, E_z \cap S(p))$ is closed, implying that $(z_i^0, z^0(i)) \in E_z \cap S(p^0)$. Thus $z_i^0 \in A_i^j(p^0, z^0(i))$ and A_i^j is an upper semicontinuous correspondence in p and $z(i)$. Upper semicontinuity of B_i^j can be shown analogously.

Let k be the integer with $L \leqslant 2k \leqslant L + 1$. Consider all possible unordered sets of k different individuals. Denote the set of these sets as η. Define $M(p, z(i))$ as

$$M(p, z(i)) = \left\{z_i : z_i \in \bigcup_{a \in \eta} \left(\bigcup_{b \in \eta} \left(\bigcap_{j \in a} A_i^j(p, z(i)) \bigcap_{l \in b} B_i^l(p, z(i)) \right)\right)\right\}.$$

Then M is the union and intersection of upper semicontinuous correspondences and hence is upper semicontinuous (see [4, p. 113]).[8]

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[8] This is the point in the proof at which assumption T1 is required. If individuals' demands are multivalued upper semicontinuous correspondences, then the median set can be discontinuous as in the example in Fig. 4. Consider p as the price of any private good.

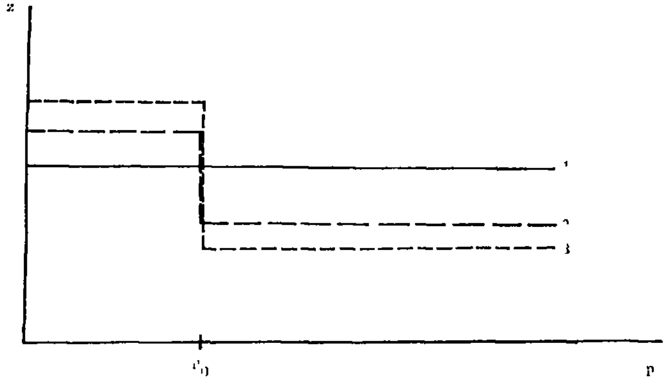

FIG. 4. A discontinuous social demand correspondence.

Curves 1, 2, and 3 are, respectively, the demands by individuals 1, 2, and 3 for some public good. For prices below and above p_0 individual 2 is median but at p_0, individual 1 is median and there is a discontinuous jump.

To complete the proof, it must be shown that $M(p, z(i)) = f_i^s(p, z(i))$. First consider that $M \subseteq f_i^s$ must hold. Take any $z_i \in M$. There must exist two sets of individuals a and b belonging to η for which $z_i \in \bigcap_{j \in a} A_i^j \bigcap_{l \in b} B_i^l$. For $j \in a$, $z_i \geq f_i^j$ and $N(z_i \geq f_i^j) \geq k \geq L/2$, while for $l \in b$, $z_i \leq f_i^l$ and $N(z_i \leq f_i^l) \geq k \geq L/2$. Then $z_i \in f_i^s$. Next it can be shown that $M \supseteq f_i^s$. Take any $z_i \in f_i^s$. Then $N(z_i \geq f_i^j) \geq L/2$ and $N(z_i \leq f_i^l) \geq L/2$. Take k individuals from the set with $z_i \geq f_i^j$ and form a and take k from the set $z_i \leq f_i^j$ and form b. For these a and b, $z_i \in \bigcap_{j \in a} A_i^j \bigcap_{l \in b} B_i^l$ and $z_i \in M$. Thus, $M = f_i^s$ is shown and f_i^s is upper semicontinuous. Q.E.D.

From Lemma's I and III come the demand relationships which will form the basis of the fixed point mapping. These demand correspondences can be summarized as

$$x = h(p, z) = \sum_{j=1}^{L} h^j(p, z),$$

$$z' = f^s(p, z) = (f_i^s(p, z(i))).$$

As summation and composition mappings, these satisfy, respectively, the various properties of Lemmas I and III. Two difficulties with these demand correspondences exist: (i) upper semicontinuity holds only for prices for which $I^j(p) = 0$, all j; (ii) they are defined only on $E_z \cap S(p)$ and not on the consumption set E_z. These problems can be dealt with by extending the mappings in two ways. The first difficulty can occur only if $p_i = 0$, $i = 1, n$, for, by assumption VI, all individuals have positive initial endowment of each private commodity. Upper semicontinuity holds on a dense subset of the price simplex, and the mappings can therefore be extended so that they are upper semicontinuous over the entire simplex, as is done, for example, in Nikaido [18, Section 16.4]. The equilibrium will be shown to require a positive price of some private good and hence cannot be on the extended portion of the correspondence.

For (ii), consider the following mapping g from E_z into $E_z \cap S(p)$:

$$g(p, z) = \{g: g \in E_z \cap S(p) \text{ and } d(g, z) \leqslant d(g', z), \forall g' \in E_z \cap S(p)\},$$

where $d(g, z)$ is the Euclidean distance between points g and z. The mapping $g(p, z)$ is nonnull, single valued, and continuous with $z = g(p, z)$ iff $z \in E_z \cap S(p)$. Then $h(p, g(p, z))$ and $f^s(p, g(p, z))$ are defined for all points in E_z, upper semicontinuous in p and z, compact and convex at each domain point and are homogeneous of degree zero in prices. At equilibrium it will be shown that $z = g(p, z)$ and hence is not on the extended part of the demand correspondences.

Next, the domain of the mapping used to show existence of an equilibrium is specified. Public goods must lie in the public good consumption set E_z. For private goods take

$$E_x' = \{x : 0 \leq x_i \leq L \cdot k\}.$$

This is the summation of the individual private good consumption sets. On the production side, let E_y be a feasible production set, bounded in a manner similar to E_x, and let

$$E_w = \left\{w: w = y + \sum_{j=1}^{L} \bar{x}^j, \forall y \in E_y\right\}.$$

For prices, the unit price simplex is utilized:

$$P = \left\{p: \sum_{i=1}^{n+m} p_i = 1, p_i \geq 0\right\}.$$

Then set A is the domain of the mapping where

$$A = E_z \chi E_x' \chi E_w \chi P.$$

A, as the Cartesian product of compact convex sets is compact and convex. For any $a \in A$, $a = (z, x, w, p) \in R^{3(n+m)}$. Define excess demands

$$\begin{aligned} e_i &= x_i - w_i, & i &= 1, n, \\ e_i &= z_{i-n} - w_i, & i &= n + 1, n + m. \end{aligned}$$

Consider mapping $\Gamma: A \to 2^{R^{3(n+m)}}$:

$$\begin{aligned} z' &= f^s(p, g(p, z)), \\ x' &= h(p, g(p, z)), \\ w' &= \sum_{j=1}^{L} \bar{x}^j + y(p), \end{aligned}$$

$$p_i' = (p_i + \max[0, e_i]) \Big/ \left(1 + \sum_{j=1}^{n+m} \max[0, e_j]\right), \qquad i = 1, n + m.$$

Lemma IV. *There exists an a^* such that $a^* \in \Gamma(a^*)$.*

Proof. Set A is a nonempty compact convex set. For each $a \in A$, $\Gamma(a)$ has been shown to be nonempty, convex, and upper semicontinuous. By construction of f_i^s, if $z_i' \in f_i^s$ then $z_i' \in \text{proj}_i\{E_z \cap S(p)\}$ so that $0 \leq z_i' \leq k$, for all i, and $z' \in E_z$. By construction, $0 \leq h_i^j \leq k$ so that $0 \leq h_j \leq L \cdot k$ and $x' \in E_x'$. By assumption, for any $y \in y(p)$, $y \in E_y$ so that $y + \sum_{j=1}^{L} \bar{x}^j \in E_w$. Since $\sum_{i=1}^{n+m} p_i = 1$, then $\sum_{i=1}^{n+m} p_i' = 1$ and $p' \in P$. Thus, for any $a' \in \Gamma(a)$,

$a' \in A$. Γ thus satisfies the conditions of the Kakutani fixed point theorem and there must exist the required a^*. Q.E.D.

THEOREM. *Under the given assumptions, there exists a nontrivial general political equilibrium.*

Proof. The fixed point $a^* = (z^*, x^*, w^*, p^*)$ must be shown to satisfy the conditions of Definition VI. Part (a) of this definition requires that z^* be a nontrivial voting equilibrium. This requires first that z^* lie within the modified social policy space, that is $z^* \in E_z \cap S(p^*)$. Since z^* is a component of a fixed point, $z^* \in f^s(p^*, g^*)$, where $g^* = g(p^*, z^*)$. Assume $z^* \neq g^*$. By construction, $(z_i^*, g^*(i)) \in E_z \cap S(p^*)$, all i. Consider

$$z'' = \sum_{i=1}^{m} [z_i^*, g^*(i)]/m = [z^*/m + (m-1)\, g^*/m].$$

By convexity of $E_z \cap S(p^*)$, $z'' \in E_z \cap S(p^*)$ with $d(z'', z^*) < d(g^*, z^*)$. This contradicts the definition of $g(p, z)$. Therefore, $z^* = g^*$ must hold. By construction, $g^* \in E_z \cap S(p^*)$ and z^* is feasible as required by Definition VIa(i).

Second, to satisfy Definition VIa it must be shown that z^* defeats by majority voting any other point in its voting feasible set. With the voting vectors parallel to the coordinate axes, the voting feasible set is those points $z_i^* + \lambda$, for any i and some λ, with $(z_i^* + \lambda, z^*(i)\} \in E_z \cap S(p^*)$. By definition of f^s, since $z_i^* \in f_i^s$ then $N(f_i^j \leqslant z_i^* < z_i^* + \lambda) \geqslant L/2$, if $\lambda > 0$. Consider any individual j in this set. For some $1 \geqslant \alpha > 0$, $(z_i^*, z^*(i)) = \alpha(f_i^j, z^*(i)) + (1 - \alpha)(z_i^* + \lambda, z^*(i))$. In terms of the derived utility function $H_p^j(z)$, since f_i^j is maximal, $H_p^j(f_i^j, z^*(i)) > H_p^j(z_i^* + \lambda, z^*(i))$, and since $H_p^j(z)$ is strictly quasi-concave, $H_p^j(z_i^*, z^*(i)) > H_p^j(z_i^* + \lambda, z^*(i))$. Thus, $N(z^* P_j(z_i^* + \lambda, z^*(i))) \geqslant L/2 \geqslant N((z_i^* + \lambda, z^*(i))\, P_j z^*)$ and z_i^* defeats all points with $\lambda > 0$. A similar proof follows with $\lambda < 0$. Since this is true for all i, z^* is a voting equilibrium and Definition VIa(ii) is satisfied.

Third, to satisfy Definition VIa, z^* must be shown to be nontrivial so that $v(z^*) - \{z^*\} \neq \phi$. For any i, either (1) $z_i^* > 0$ or (2) $z_i^* = 0$. If (1), then for some $\lambda < 0$, $z_i^* + \lambda$ lies along a vector and is within the social policy space. If (2) and $p_{n+i}^* > 0$, then excess demand $e_i^* < 0$ since $y_{n+i}^* > 0$ from assumption TIV. This is shown to be inconsistent in the next part of the proof. With $p_{n+i}^* = 0$, then any $0 < \lambda \leqslant k$ lies along a vector and is within the feasible space. Thus, there must exist a feasible λ for both (1) and (2) and Definition VIa(iii) is shown.

Part (b) of Definition VI requires that y^* be on the supply correspondence, that for each j, x^{*j} be maximal, and that supply equal demand in all markets. By construction, $y^* \in y(p^*)$ and $x^* = h(p^*, z^*)$ and thus the first requirements are satisfied. To have demand equal supply, excess demand e_i^* must satisfy $e_i^* \leq 0$ and $p_i^* e_i^* = 0$, $i = 1, n + m$. Let $e^* = \sum_{i=1}^{n+m} \max(o, e_i^*)$. Then $p_i^* = (p_i^* + \max(o, e_i^*))/(1 + e^*)$, all i, in order for p_i^* to be a component of the fixed point. Multiplying by $1 + e^*$ and subtracting p_i^* gives $e^* p_i^* = \max(o, e_i^*)$, all i, which multiplied by e_i^* and summed over all i gives $e^* \sum_{i=1}^{n+m} p_i^* e_i^* = \sum_{i=1}^{n+m} (\max(o, e_i^*))^2 \geq 0$. Summing the budget constraints of the individuals, each of which, from nonsatiation, has $p_x^* x^{*j} + t_j^* p_z^* z^* = p_x^* \bar{x}^j + \psi^j(p^*)$, using the facts that $\sum_{j=1}^{L} t_j^* = 1$ and $\sum_{j=1}^{L} \psi^j(p^*) = p^* y^*$, gives $p_x^* x^* + p_z^* z^* = p_x^* \sum_{j=1}^{L} \bar{x}^j + p^* y^*$, or equivalently, $\sum_{i=1}^{n+m} p_i^* e_i^* = 0$, which is Walras' law. Then

$$\sum_{i=1}^{n+m} (\max(0, e_i^*))^2 = 0 \qquad \text{and} \qquad e_i^* \leq 0 \qquad \text{all } i.$$

Since $p_i^* e_i^* \leq 0$, all i, and $\sum_{i=1}^{n+m} p_i^* e_i^* = 0$, then $p_i^* e_i^* = 0$, all i. For $i > n$, $e_i^* = 0$ when $p_i^* = 0$, since $w_i = 0$ from TIV. Thus Definition VIb is shown.

The mappings had been extended over prices at which $\min\{I^j(p)/t_j\} = 0$. Completing the proof requires showing that $I^j(p^*) > 0$ all j. If $I^j(p^*) = 0$, then $p_i^* = 0$, $i = 1, n$, from assumption VI and $z_j^* = 0$, all j, from Definition V. There must exist a $p_{n+k}^* > 0$, since $\sum_{i=1}^{n+m} p_i^* = 1$. Then from assumption TIV, $y_{n+k}^* > 0$ so that $e_{n+k}^* = z_k^* - y_{n+k}^* < 0$. Thus $p_{n+k}^* e_{n+k}^* < 0$, contradicting $p_i^* e_i^* = 0$, all i. Hence $p_i^* > 0$, some $i \leq n$, and the fixed point is on the unextended part of the mapping. Also, the bounds on the consumption and production sets can be chosen large enough so that, in equilibrium, they are not binding.

All parts of Definition VI are satisfied and the fixed point is a nontrivial general political equilibrium.

The above proof was for voting vectors parallel to the coordinate axis. For nonparallel vectors, consider the transformation of variables from z into $\bar{z}$ defined by $z = V\bar{z}$ or $\bar{z} = (V^{-1})z$, where V is the nonsingular matrix of voting vectors. The consumption set and utility functions are transformed similarly with $E_{\bar{z}} = \{z : 0 \leq (V\bar{z})_i \leq k\}$ and with $\bar{U}^j(x, \bar{z}) = U^j(x, V\bar{z}) = U^j(x, z)$. $E_{\bar{z}}$ is nonempty, compact, and convex and $\bar{U}^j(x, \bar{z})$ is continuous and strictly quasi-concave. The budget set for each individual is $p_x x^j + t_j p_{\bar{z}} \bar{z} \leq I^j(p)$, where $p_{\bar{z}} = p_z V$. From these the social policy space and the set $S(p)$ can also be defined in the transformed space. Lemmas I–III follow as in the text, replacing the original variable and spaces with these transformed ones. For the fixed point mapping of Lemma IV, the public good consumption space $E_{\bar{z}}$ must be replaced by $\bar{E}_{\bar{z}} = \prod_{i=1}^{m} \text{proj}_i\, E_{\bar{z}}$, since $f^s = (f_i^s)$ need not lie in $E_{\bar{z}}$. To complete the mapping, the domain of $g(p, z)$ becomes $\bar{E}_{\bar{z}}$ and the range $E_{\bar{z}} \cap S(p)$.

The proof that the fixed point of Lemma IV is a general political equilibrium follows with only slight modification. Note that choice along vector v^i

is the same as choice over $\bar{z}_i$ with $\bar{z}(i)$ held fixed. Excess demand for public goods will be defined by $e_{n+i} = \{V\bar{z}\}_i - w_{n+i}$. The proof that the fixed point satisfies Definition VIa(iii) must use the vector with $r_k^j > 0$, some k, and $r_h^j = 0$, $h \neq k$, specified in TII(b), as the one along which some choice is available.

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Choice among Distributions*

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I. INTRODUCTION

In this paper we present a generalization of the stochastic dominance results of Hadar and Russell [3], Hanoch and Levy [4], and Meyer [7]. As an extension of the theory of stochastic dominance, this work is important for two reasons. First, it develops new results by defining a more general form of stochastic dominance and relating it to unanimous preference by certain well-defined groups of agents. Second, it presents a general framework which allows most of the existing work in stochastic dominance to be viewed as special cases.

From an applied-empirical standpoint the results presented here are even more important in that the applicability of the stochastic dominance concept to empirical research is greatly increased. No longer is stochastic dominance in theory an ideal way of ranking distributions while in practice a poor way, because of the large number of distributions which are unrankable. Besides making stochastic dominance a more useful tool for ranking distributions, these results also open new areas for empirical research. Examples of possible areas of application will be discussed following the presentation of the theory. First, we discuss the notation used here, the problem stochastic dominance concerns itself with, and the general framework used to study this problem.

II. ASSUMPTIONS AND NOTATION

The general problem studied under the heading stochastic dominance is that of finding necessary and sufficient conditions on cumulative distributions $F(x)$ and $G(x)$ for $F(x)$ to be preferred or indifferent to $G(x)$ by all agents in a particular group of agents. Since we will examine this problem for various

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groups of agents, we first discuss how the particular groups of agents are to be described.

We assume that all agents are such that their preferences are represented by an expected utility function $u(x)$ which is increasing and twice differentiable. It would seem natural then to describe groups of agents in terms of properties of their utility functions. This is what has been done in the past work concerning stochastic dominance. However, for the purposes here we do not find this method of characterizing groups of agents very convenient. This is because $u(x)$ is not a unique representation of a given set of preferences. In fact, it is well known that any positive linear transformation of $u(x)$ also represents those same preferences. Thus, to restrict the group of agents being considered by placing restrictions on $u(x)$, one must do so using restrictions which, if met by $u(x)$, are also met by all positive linear transformations of $u(x)$. The existing work in stochastic dominance does this by using restrictions that specify the sign of the second and third derivatives of $u(x)$. These of course are unaltered by positive linear transformations of $u(x)$. Many possible restrictions on $u(x)$, such as $u(x) \geq k \ \forall x$ for some constant k, do not define groups of agents, since whether an agent belongs to the group or not depends on the particular representation of his preferences being used.

Because of the above difficulty, instead of using restrictions on $u(x)$ to specify groups of agents, we will use restrictions on $r(x)$, where $r(x)$ is defined by

$$r(x) = -u''(x)/u'(x).$$

Pratt [9] has shown that $r(x)$ represents an agent's preferences, and more importantly, it does so uniquely. It is this uniqueness that makes $r(x)$ a more desirable representation of preferences for the purposes here, since restrictions on $r(x)$ now correspond directly to restrictions on preferences. Thus, they can be used more easily to define groups of agents.

We will describe groups of agents in the following manner. Let $U(r_1(x), r_2(x))$ be the set of agents with preferences represented by $r(x)$ satisfying

$$r_1(x) \leqslant r(x) \leqslant r_2(x) \qquad \forall x$$

for given functions $r_1(x)$ and $r_2(x)$. Alternatively in terms of expected utility functions $u(x)$, $U(r_1(x), r_2(x))$ can be defined as the group of agents with expected utility functions $u(x)$ satisfying

$$r_1(x) \leqslant -u''(x)/u'(x) \leqslant r_2(x) \qquad \forall x.$$

Groups of agents of this general form will be used in examining the stochastic dominance question.

The function $r(x)$, used here as a unique representation of an agent's preferences, has been interpreted by both Pratt [9] and Arrow [1] as a measure of an agent's absolute aversion to risk. Thus, our restrictions on $r(x)$ can be viewed as specifying a lower and upper bound on the degree of risk aversion for the agents in the set being considered. It is often convenient to use this interpretation of $r(x)$ when describing the results presented in the pages that follow and we will do so.

Given the above general description of groups of agents, we now turn to a description of their objects of choice. We assume that agents choose between cumulative distribution functions $F(x)$ and $G(x)$ which have points of increase in a bounded interval taken for simplicity to be $[0, 1]$. Thus, $F(0) = G(0) = 0$ and $F(1) = G(1) = 1$. The expected utility assumption then states that $F(x)$ is preferred or indifferent to $G(x)$ by an agent with utility function $u(x)$ if and only if

$$\int_0^1 u(x)\, dF(x) \geqslant \int_0^1 u(x)\, dG(x),$$

or equivalently,

$$\int_0^1 [G(x) - F(x)]\, u'(x)\, dx \geqslant 0.$$

All integration is Lebesque-Stieltjes integration.

In short then, the general problem we will consider here is as follows. For any two functions $r_1(x)$ and $r_2(x)$ find necessary and sufficient conditions on cumulative distributions $F(x)$ and $G(x)$, for $F(x)$ to be preferred or indifferent to $G(x)$ by all agents in the class $U(r_1(x), r_2(x))$. Using this general framework we will now review the existing work concerning this problem.

III. LITERATURE REVIEW

All of the existing work addressed to the above problem solves it for a specific special case. Much of the work is presented under the heading stochastic dominance, although Hanoch and Levy used the heading efficiency criteria. Hadar and Russell presented the concept of first-degree stochastic dominance and used it to solve the problem for the special case where $r_1(x) = -\infty$ and $r_2(x) = +\infty$. That is, if no restrictions on agents' preferences other than those imposed by assuming $u(x)$ is increasing and twice differentiable are specified, then first-degree stochastic dominance gives necessary and sufficient conditions on $F(x)$ and $G(x)$ for $F(x)$ to be preferred or indifferent to $G(x)$ by all such agents. Formally, Hadar and Russell proved the following theorem.

THEOREM 1. *$F(x)$ is preferred or indifferent to $G(x)$ by all agents in the set $U(-\infty, +\infty)$ if and only if*

$$[G(x) - F(x)] \geq 0 \qquad \forall x \in [0, 1].$$

Hadar and Russell also solved the problem for another important special case, namely, when $r_1(x) = 0$ and $r_2(x) = +\infty$. The conditions on $F(x)$ and $G(x)$ for $F(x)$ to be preferred or indifferent to $G(x)$ by all agents in this class are given as follows.

THEOREM 2. *$F(x)$ is preferred or indifferent to $G(x)$ by all agents in the set $U(0, +\infty)$ if and only if*

$$\int_0^y [G(x) - F(x)]\, dx \geq 0 \qquad \forall y \in [0, 1].$$

These conditions on $F(x)$ and $G(x)$ are what Hadar and Russell define as second-degree stochastic dominance of $F(x)$ over $G(x)$. This particular group of agents, $U(0, +\infty)$, is an important group since it is the set of all risk averse agents.

Recently, Meyer [7] has solved the problem for two special cases of a more general nature. In the first case, by defining stochastic dominance with respect to a function, Meyer has solved the problem when $r_1(x)$ is any function and $r_2(x) = +\infty$. The following theorem states this result.

THEOREM 3. *For any $r_1(x) = -u_1''(x)/u_1'(x)$, $F(x)$ is preferred or indifferent to $G(x)$ by all agents in the set $U(r_1(x), +\infty)$ if and only if*

$$\int_0^y [G(x) - F(x)]\, u_1'(x)\, dx \geq 0 \qquad \forall y \in [0, 1].$$

Thus, Meyer has solved the problem for the case when only a lower bound on the risk aversion of the group of agents is specified. In the same paper, Meyer solves the problem for the symmetric case when only an upper bound on the risk aversion of the group of agents is specified; that is, for the case when $r_1(x) = -\infty$ and $r_2(x)$ is any function.

THEOREM 4. *For any $r_2(x) = -u_2''(x)/u_2'(x)$, $F(x)$ is preferred or indifferent to $G(x)$ by all agents in the set $U(-\infty, r_2(x))$ if and only if*

$$\int_y^1 [G(x) - F(x)]\, u_2'(x)\, dx \geq 0 \qquad \forall y \in [0, 1].$$

Thus, the problem has been solved when only an upper bound or a lower bound oh the risk aversion of the group of agents is specified.

This summarizes the literature. The general problem of finding necessary and sufficient conditions on $F(x)$ and $G(x)$ for $F(x)$ to be preferred or indifferent to $G(x)$ by all agents in the set $U(r_1(x), r_2(x))$ for general functions $r_1(x)$ and $r_2(x)$ has not been solved. This is the problem studied here in this paper.

IV. METHOD OF SOLUTION

We will solve the problem just described by finding that the utility function $u(x)$ which satisfies

$$r_1(x) \leq -u''(x)/u'(x) \leq r_2(x) \qquad \forall x \in [0, 1] \tag{1}$$

and minimizes

$$\int_0^1 [G(x) - F(x)]\, u'(x)\, dx. \tag{2}$$

Since this last expression equals the expected utility from $F(x)$ minus the expected utility from $G(x)$, we know that if we minimize it over a set of agents and the minimum is greater than or equal to zero then the set of agents unanimously prefer or are indifferent between $F(x)$ and $G(x)$. We also know that if the minimum is less than zero then the set of agents is not unanimous in choosing $F(x)$ over $G(x)$. Thus, the set of agents $U(r_1(x), r_2(x))$ unanimously prefers or is indifferent between $F(x)$ and $G(x)$ if and only if the minimum of (2) subject to (1) is greater than or equal to zero. We can then solve our originally posed problem by finding that $u(x)$ which satisfies (1) and minimizes (2), and checking to see if this minimum is nonnegative or not.

A problem that must be taken care of is the fact that the minimum will not exist because $u(x)$ is unique only to a positive linear transformation. This implies that (2) is unique only to a positive multiple. Since this does not affect the nonnegativeness of (2), we can without loss of generality remove this indeterminacy by assuming that $u'(0) = 1$.

Our problem has then been transformed into the following equivalent one. For any $r_1(x)$ and $r_2(x)$ find that $u(x)$ which satisfies both

$$r_1(x) \leq -u''(x)/u'(x) \leq r_2(x) \qquad \forall x \in [0, 1]$$

and $u'(0) = 1$ and minimizes

$$\int_0^1 [G(x) - F(x)]\, u'(x)\, dx,$$

then check to see if this minimum is nonnegative or not. It is nonnegative if and only if all agents in the set $U(r_1(x), r_2(x))$ are unanimous in choosing $F(x)$ over $G(x)$.

V. PROBLEM SETTING AND SOLUTION

A possible setting for the above problem is in the optimal control framework. We will reformulate the problem as a maximization using the specific optimal control framework found in Arrow and Kurz [2, pp. 39–45].

PROBLEM. Using control variable $-u''(x)/u'(x)$, maximize the objective function

$$-\int_0^1 [G(x) - F(x)]\, u'(x)\, dx \tag{3}$$

subject to: the equation of motion

$$(u'(x))' = u'(x)[u''(x)/u'(x)]; \tag{4}$$

the control constraints

$$[-u''(x)/u'(x)] - r_1(x) \geqslant 0, \tag{5}$$

$$[+u''(x)/u'(x)] + r_2(x) \geqslant 0; \tag{6}$$

and the initial condition

$$u'(0) = 1. \tag{7}$$

This is a very straightforward problem in control theory. Arrow and Kurz [2, p. 41] give necessary conditions for $-u_0''(X)/u_0'(x)$ to be a solution to problems of this exact format. Since the Hamiltonian is linear in the state variable, these conditions are sufficient as well. However, to apply the Arrow and Kurz theorem we must assume that $F(x)$ and $G(x)$ are continuous distribution functions. This is to ensure that the integrand of the objective function is continuous in all variables. Hestenes [5, pp. 254–270] proves several theorems which are of a more general nature and include this problem format as a special case. Proving these theorems Hestenes assumes that the integrand of the objective function is continuous and differentiable in the state variable only. Thus, the properties of a solution given in Arrow and Kurz and used below are valid for both discrete and continuous distribution functions. Rather than write out these properties of a solution we will present only those of interest to us here.

The first property of interest to us is that the optimal control $-u_0''(x)/u_0'(x)$ maximizes the Hamiltonian H at all points x subject to constraints (5) and (6). The Hamiltonian is given by

$$H = -[G(x) - F(x)]\, u'(x) + p(x)\, u'(x)[u''(x)/u'(x)],$$

where $p(x)$ is the costate variable. Thus, $-u''(x)/u'(x)$ must be at its maximum if $p(x)\,u'(x)$ is negative and at its minimum if $p(x)\,u'(x)$ is positive. That is,

$$-\frac{u_0''(x)}{u_0'(x)} = \begin{cases} r_1(x) & \text{if } p(x)\,u'(x) > 0, \\ r_2(x) & \text{if } p(x)\,u'(x) < 0. \end{cases}$$

A second property of a solution that is of interest to us defines the costate variable $p(x)$: $p'(x) = -\partial L/[\partial u'(x)]$, evaluated at the optimal path where

$$L = H + q_1\left(-\frac{u''(x)}{u'(x)} - r_1(x)\right) + q_2\left(\frac{u''(x)}{u'(x)} + r_2(x)\right).$$

Since the constraints only involve the control variable,

$$\partial L/[\partial u'(x)] = \partial H/[\partial u'(x)].$$

Therefore $p'(x) = +[G(x) - F(x)] - p(x)[u_0''(x)/u_0'(x)]$.

Now

$$\begin{aligned} \frac{d[p(x)\,u_0'(x)]}{dx} &= p'(x)\,u_0'(x) + p(x)\,u_0''(x) \\ &= \left[[G(x) - F(x)] - p(x)\,\frac{u_0''(x)}{u_0'(x)}\right] u_0'(x) + p(x)\,u_0''(x) \\ &= [G(x) - F(x)]\,u_0'(x). \end{aligned}$$

Thus,

$$\int_x^1 \frac{d(p(x)\,u_0'(x))}{dx}\,dx = \int_x^1 [G(x) - F(x)]\,u_0'(x)\,dx.$$

Using the transversality condition $p(1)\,x(1) = 0$ we see that

$$-p(x)\,u_0'(x) = +\int_x^1 [G(x) - F(x)]\,u_0'(x)\,dx$$

or

$$p(x)\,u_0'(x) = -\int_x^1 [G(x) - F(x)]\,u_0'(x)\,dx.$$

Therefore we have proved the following theorem.

THEOREM 5. *An optimal control* $-u_0''(x)/u_0'(x)$ *which maximizes* $-\int_0^1 [G(x) - F(x)]\, u'(x)\, dx$ *subject to* $r_1(x) \leqslant [-u''(x)/u'(x)] \leqslant r_2(x)$ *and* $u'(0) = 1$ *is given by*

$$\frac{-u_0''(x)}{u_0'(x)} = \begin{cases} r_1(x) & \text{if } \int_x^1 [G(y) - F(y)]\, u_0'(y)\, dy < 0, \\ r_2(x) & \text{if } \int_x^1 [G(y) - F(y)]\, u_0'(y)\, dy \geqslant 0. \end{cases}$$

This theorem says that at any point x we choose the control variable to be either its maximum or its minimum depending only on the sign of the objective function from point x forward to 1 using the optimal control over the interval $[x, 1]$. Thus, if we know the solution from x forward we can determine the solution at x. This type of solution involving calculation of the solution at later points first is typical in dynamic programming problems.

Since $-u_0''(x)/u_0'(x)$ completely represents a given set of preferences, our problem has been solved. From the optimal control $-u_0''(x)/u_0'(x)$ we can find $u_0'(x)$ and check if the expected utility from distribution $F(x)$ is greater than or equal to the expected utility from $G(x)$ for this agent, and thus determine whether $F(x)$ is preferred or indifferent to $G(x)$ by all agents in the set $U[r_1(x), r_2(x)]$. In theory the problem posed has been solved. However, the solution is not in closed form and we have been unable to obtain a closed form solution. From an applied standpoint this is not discouraging, since the solution for a given $F(x)$, $G(x)$, $r_1(x)$, and $r_2(x)$ can be calculated with comparative ease. This will be discussed in the next session.

VI. CALCULATING THE SOLUTION

The problem of finding necessary and sufficient conditions on $F(x)$ and $G(x)$ for $F(x)$ to be preferred or indifferent to $G(x)$ by all agents in the set $U[r_1(x), r_2(x)]$ was solved in the last section. The solution, however, was in the form of a rule for calculating the solution. Thus, its importance for applied work depends heavily on whether or not this calculation can actually be performed. We will show here that it can be performed quite easily and demonstrate how it can be done.

According to Theorem 5, in order to calculate the solution at any point x the optimal solution from x forward must be known. Thus, in order to get the process started one must be able to get the solution in some interval $[x, 1]$ and then use the rule outlined in Theorem 5 to obtain the solution in the rest of the interval $[0,1]$. Fortunately, obtaining the solution in some interval $[x, 1]$ is not difficult. To see how to do this consider the following special case. Suppose the distribution $G(x)$ and $F(x)$ are such that $G(x) - F(x)$ is always nonnegative. Then

$$\int_y^1 [G(x) - F(x)]\, u'(x) \geq 0 \qquad \forall y \in [0, 1]$$

for any $u'(x)$ we consider. Thus, the optimal control for this particular $F(x)$ and $G(x)$ is to choose $-u''(x)/u'(x)$ equal to its maximum possible; i.e., $-u_0''(x)/u_0'(x)$ $r_2(x)$ at all points x. Similarly, if $F(x)$ and $G(x)$ are such that $G(x) - F(x)$ is always nonpositive then an optimal choice is to set $-u''(x)/u'(x)$ equal to its minimum throughout, $r_1(x)$. So if $G(x) - F(x)$ does not change sign the optimal solution is easily given.

Now suppose that $G(x) - F(x)$ changes sign a finite number of times. Then we know that for some x_0, $G(x) - F(x)$ does not change sign in the interval $[x_0, 1]$. Thus, the optimal solution in $[x_0, 1]$ is given by the above, and once the solution in $[x_0, 1]$ is known, the solution in $[0, x_0]$ can be calculated according to the rule in Theorem 5. Therefore, for any distributions $F(x)$ and $G(x)$, such that $G(x) - F(x)$ changes sign a finite number of times, we can calculate the solution. Since this restriction is not a major one, most important problems can be solved by this calculation method.

To check the time and expense involved in performing the calculations we have written a computer program to obtain the solution. We checked the program using distribution of rate of return for 34 mutual funds and the Dow Jones average. We found that the computer can compare each of the distributions for the mutual funds with the distribution for the Dow Jones average in less than 5 sec. This was for any of the $r_1(x)$ and $r_2(x)$ that we specified. The results of these comparisons are interesting in their own right and are reported elsewhere (see [8]).

In summary, we have given a method for checking whether distribution $F(x)$ is preferred or indifferent to $G(x)$ by all agents in the class $U(r_1(x), r_2(x))$ for any $r_1(x)$ and $r_2(x)$ and for any distributions $F(x)$ and $G(x)$ such that $G(x) - F(x)$ changes sign a finite number of times. This result has many important applications, some of which will be discussed in the next section.

VII. APPLICATIONS

To indicate the possible applications of the results presented here, we will discuss three broad areas of use. The first of these involves the ability to answer two types of questions about a given pair of cumulative distributions $F(x)$ and $G(x)$. We can now answer questions of the form, "is $F(x)$ preferred or indifferent to $G(x)$ by all agents in the set $U(r_1(x), r_2(x))$?" This question can be answered for any $r_1(x)$ and $r_2(x)$ we care to specify. An example of this type of question is to ask whether the Dow Jones average is preferred to most or all mutual funds by a set of agents of the form $U(r_1(x), r_2(x))$. A second type of question that falls into this category is to ask what groups of agents prefer $F(x)$ to $G(x)$ for a particular $F(x)$ and $G(x)$. To partially answer this, one can systematically vary $r_1(x)$ and $r_2(x)$ trying to find an $r_1(x)$ and $r_2(x)$ such that $F(x)$ is preferred or indifferent to $G(x)$ by all agents in $U[r_1(x), r_2(x)]$.

A second area of application involves the idea of an efficiency criterion. An efficiency criterion is a way of reducing the choice set by eliminating all elements that are dominated by other elements in the set. Element A is said to dominate B if all agents being considered would choose A over B. obviously, efficiency criteria are defined relative to a given group of agents. An example of an often used efficiency criterion is mean-variance efficiency. This criterion is defined relative to the set of agents who choose between distributions on the basis of means and variances alone. Other well-known efficiency criteria are given by first-degree and second-degree stochastic dominance (see [4]). The results presented here can be viewed as giving an efficiency criterion for any group of agents of the form $U[r_1(x), r_2(x)]$. That is, using the results here one can eliminate from the choice set all elements which are dominated by some other element, dominance being defined relative to a set of the form $U[r_1(x), r_2(x)]$.

A final area of application we will discuss involves the area of decision analysis. In decision analysis expected utility theory is often called on to determine the choice to be made among distributions. Since the expected utility function $u(x)$ is generally unknown it is estimated using various techniques. Of course this estimate is actually an estimate of the decision maker's preferences and thus can be viewed as an estimate of $r(x) = -u''(x)/u'(x)$. In practice, since the decision analyst can only estimate $u(x)$ he performs a sensitivity analysis to see if changing $u(x)$ slightly might change the choice made among distributions. The theoretical foundations for this sensitivity analysis are weak. It is difficult to decide how far and in what manner one should vary $u(x)$ to be sure to correct for the possible estimation errors. The results presented here are a beginning toward making sensitivity analysis precise. When estimating the agent's preferences it should be possible to obtain upper and lower bounds on $r(x)$ similar to confidence intervals obtained on coefficients in a standard regression model. If $r_1(x)$ represents the lower bound and $r_2(x)$ represents the upper bound then they together define a class of agents $U[r_1(x), r_2(x)]$ which contains the decision maker with a certain degree of confidence. One can then check to see if this group of agents would be unanimous in the choice they would make from the alternatives available. If they are, then one can make the choice with the specified degree of confidence. If the group would not be unanimous, then one knows that the choice is sensitive to the accuracy of the estimate. Obviously much work needs to be done before a better theoretical basis is given to sensitivity analysis but we think these results are a base on which one can begin.

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Protection and Real Wages: The Neoclassical Ambiguity*

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Perhaps the most famous proposition in the modern theory of trade is the theorem expounded in the classic article by Stolper and Samuelson [8]. If imports are labor intensive in a constant-returns-to-scale two-sector economy, a tariff that raises the price of imports unambiguously raises the real wage. Neoclassical trade theory was more indefinite concerning the impact of protection on real wages. Stolper and Samuelson, in summarizing older writings on the subject, commented that, "the vast majority of writers take it as axiomatic that a calculation of effects upon real income must take into consideration the behavior of prices of commodities entering the consumer's budget." This would follow if the rise in the price of imports exceeds, in relative amount, any rise in money wages, so that wages rise in terms of some goods but not in terms of others. By contrast, in the Stolper–Samuelson model protection would raise the money wage by relatively more than any commodity price–no problem involving cost-of-living indices need arise.

The Stolper–Samuelson theorem relies on three distinctive assumptions: there are only two productive factors; both factors are perfectly mobile; and there is no joint production. It is through these assumptions that the neoclassical ambiguity is resolved. Drop any of these assumptions and cost-of-living considerations reemerge.

This paper analyzes the link between protection and real wages without the benefit of the Stolper–Samuelson assumptions. We argue that there is a presumption that protection will hurt labor. This arises from the mere fact

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that the good being protected is in excess demand, which means, from the standpoint of the whole economy, that an increase in its price must hurt someone. So stated the argument is trivial. To add economic content, we investigate the Ricardo–Viner model (as Samuelson calls it)[1] in which there are many factors but labor is the only *mobile* factor. In such a model, to argue that protection benefits labor is to argue that the industry's technological characteristics and workers' tastes are *biased* in a well-defined sense when compared to the whole economy.

I. THE CRITERION

Suppose labor, V_L, is one of many factors of production in a full employment general equilibrium competitive model with or without joint production. Suppose the price of one commodity, p_j, rises, with all other commodity prices constant. This will, in general, change the money wage rate, w_L. Letting a $\wedge$ over a variable denote relative changes ($\hat{x}$ is dx/x), define β_{Lj} as the relative rise in the money wage when p_j increases by 1%:

$$\beta_{Lj} = \hat{w}_L/\hat{p}_j . \qquad (1)$$

Turning to the consumption side, let θ_j^L represent the fraction of workers' expenditures devoted to commodity j. An increase in p_j tends to erode workers' real income by the "terms-of-trade" effect, $-\theta_j^L\hat{p}_j$.

Putting these two effects together, the *net* change in the real income of workers (y_L) is given by criterion (2):

$$\hat{y}_L = (\beta_{Lj} - \theta_j^L)\hat{p}_j . \qquad (2)$$

If β_{Lj} is a positive fraction, the impact of the price change on real wages is left in doubt. This is the neoclassical ambiguity not present in the Stolper–Samuelson model.

II. PRESUMPTION IN THE GENERAL CASE

It is illuminating first to consider the Ricardian trade model. Homogeneous labor is allocated to different activities. Let a_{Lj} be the labor requirements of one unit of good j; and assume the a_{Lj}'s are constants. If the jth good is exported, then $w = p_j/a_{Lj}$, so that $\hat{p}_j = \hat{w}$, i.e., $\beta_j = 1$; if the jth good is not produced locally, but imported instead, $\beta_j = 0$. Clearly, the Ricardian model yields an unambiguous result when criterion (2) is applied: labor always gains or loses as the price of exports or imports rises. This, of course, follows because labor is the *only* productive factor. But let us generalize. Being the only factor, labor is also the "average" or "typical" factor. This raises the natural question: If labor is the "typical" factor in a many-factor world, would it not also be true that real wages and the terms of trade would be linked in a similar way?

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[1] See Samuelson [7], and other recent treatments of this model by Jones [2, 3], Mayer [4], and Mussa [5].

A neutral assumption about tastes is that workers' preferences are biased neither in favor of nor against commodity j relative to the populace at large. That is, if θ_j represents the fraction of income spent on good j by the economy as a whole, the neutrality assumption is that θ_j^L equals θ_j.

Turning to the production side consider the impact of a rise in p_j (all other commodity prices constant) on the entire range of factor prices, w_i. Some will rise, some may fall. The *average* value of the relative changes in all factor prices would be obtained by weighting each $\hat{w}_i$ by the distributive share of that factor in the national income, α^i. That is, the weighted average of the factor price changes is

$$\sum_i \alpha^i \hat{w}_i, \qquad \text{where} \quad \alpha^i > 0, \ \sum \alpha^i = 1.$$

The concept of bias is prompted by this average: commodity j is said to be unbiased with respect to labor if the relative change in the wage rate ($\hat{w}_L$) brought about by an increase in p_j is precisely the average for the changes in all factor prices. Alternatively, commodity j is *biased towards labor* if the change in the wage rate encouraged by a rise in p_j exceeds the average for all factor prices. Clearly if p_j rises labor's *real* wage is more apt to rise the greater is j's bias towards labor. In the two-commodity, two-factor Stolper–Samuelson world, if j is the labor-intensive good it is so biased towards labor that β_{Lj} exceeds unity and the real wage *must* rise. Such a strong bias need not be found in a more general neoclassical setting, but it becomes relevant to ask if there is any presumption as to the real wage in a "typical" case—in which commodity j is neither biased towards nor against labor.

The key property required to investigate the consequence of unbiasedness is satisfied for a wide variety of general equilibrium models. It states that if commodity and factor prices are disturbed, and if α_h denotes the fraction of the national income represented by the production of commodity h and α^i the distributive share of factor i in the national income, then

$$\sum_h \alpha_h \hat{p}_h = \sum_i \alpha^i \hat{w}_i. \tag{3}$$

That is, the output-share weighted average of relative changes in commodity prices equals the distributive share weighted average of relative factor price changes. This is a proposition that follows from the competitive profit conditions: For each activity undertaken in the economy the costs of running the activity must equal the value of all outputs in the activity.[2] Relationship (3) follows by differentiating the competitive profit conditions and adding over all sectors.[3] Note that it guarantees that a doubling of all factor prices would be consistent with a doubling of all commodity prices—a property of the general neoclassical model.

Suppose, now, that p_j is the only commodity price to rise. The left-hand side of (3) reduces to $\alpha_j \hat{p}_j$. If commodity j is unbiased with respect to labor, the relative change in the wage rate is, by definition, given by the right-hand side of (3). That is, in the unbiased case,

$$\beta_{Lj} = \alpha_j . \tag{4}$$

Reconsider criterion (2). If workers' tastes are neutral, the share of wages devoted to consumption of commodity j, θ_j^L, is the average for the entire community, θ_j. On the production side unbiasedness requires β_{Lj} to equal α_j. But $(\theta_j - \alpha_j)$ is the fraction of the national income (Y) represented by *imports* of commodity j (M_j). That is, criterion (2) for the neutral case states:

$$\hat{y}_L = -(p_j M_j / Y) \cdot \hat{p}_j . \tag{5}$$

From this it follows that:

> A rise in the price of any commodity that is unbiased with respect to labor raises workers' real income if and only if that commodity is exported. An attempt to raise the domestic price of an import good reduces the real wage.

Apply this to tariff theory by considering the case in which the country is small in the sense that it is unable to influence its world terms of trade. Then if good j is imported, a tariff serves to raise the domestic price of commodity j by the full amount of the tariff. If commodity j is exported, an export tax would *lower* the domestic price by the amount of the tax. Until a later section ignore the question of the proceeds of the tariff revenue. (This is in line with the Stolper–Samuelson discussion.) Then in this neutral case workers are against any interference with free trade—a tariff on any imported commodity or an export tax on any commodity being exported unambiguously lowers workers' real income. Workers would press for protection of an importable only if that good is sufficiently biased towards labor.

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[2] Activities which are not undertaken because unit costs exceed value cause no problem.

[3] A crucial ingredient in the formal proof is the assumption that least-cost techniques are chosen.

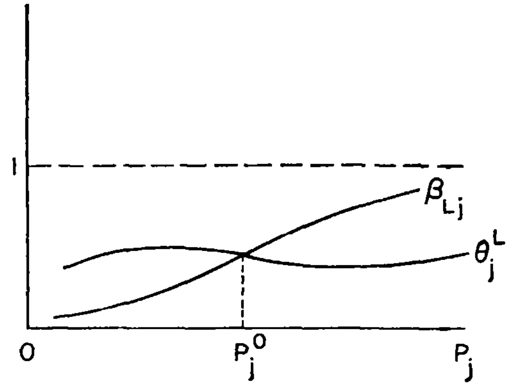

FIG. 1. For any p_j, the change in worker's real income corresponding to a rise in p_j is $(\beta_{Lj} - \theta_j^L)\hat{p}_j$. In the "neutral" case the curves intersect at p_j^0, the autarky price at which j is neither imported nor exported. Commodity j is imported for $p_j < p_j^0$.

Figure 1 illustrates relationship (2) for the unbiased case. The β_{Lj} curve shows that the higher is j's price (and output), the greater impact would any price rise have on money wages. The zero-unity limits follow from the bounds on production and consumption shares. No special assumptions as to demand elasticities of workers for commodity j are made, save that consumption of all commodities is always positive and the two curves only intersect once.[4] The unbiased case is characterized by the coincidence of the intersection point, p_j^0, with the autarky price of j. For $p_j > p_j^0$ in Fig. 1, production of commodity j would rise and consumption fall; commodity j would be exported. Any increase in p_j in this range would help workers. If commodity j is imported, the value of p_j must be to the left of p_j^0 in Fig. 1. Any attempt to raise p_j would hurt workers more via the increase in the cost of living than they would gain through a higher money wage.

Workers gain from a price rise through any increase in the money wage, but have this gain offset by a consumption loss. Neutrality links the wage gain to the share of the commodity in production and the consumption loss to the share in consumption. Therefore the direction of trade, whether consumption exceeds or falls short of production, becomes the crucial ingredient in the criterion for the change in the *real wage*.

III. AN APPLICATION TO THE RICARDO–VINER MODEL

Not all general equilibrium models of production admit of the possibility of commodities unbiased with respect to labor (or any other factor). The

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[4] For the "neutral" case discussed here, the single intersection would follow from the standard assumption that taste patterns for the community can be represented by a set of nonintersecting community indifference curves.

Heckscher–Ohlin model with two factors and two commodities does not, although generalizing to higher-dimensional cases easily could. The Ricardo–Viner model of trade does allow a lack of bias of some commodity (or more than one) with respect to labor. It may be instructive to develop formal expressions for β_{Lj} in such a model to probe more deeply into the characteristics of a commodity that would lead to neutrality or bias.

Assume the economy produces n commodities under competitive conditions and constant returns to scale. Labor, the mobile factor, earns the same return in all industries. Each sector also employs a nonlabor resource (call it land) which is immobile from one sector to the others. Assume that the price of commodity j rises, all other prices constant. What is the impact on the money wage? β_{Lj}, the answer to this question, must lie between zero and unity in the Ricardo–Viner model.

An exact expression for β_{Lj} has been derived in Jones [2], Mussa [5], etc., and is merely noted here. The change in the return to labor, $\hat{w}_L$, is a positive weighted average of all commodity price changes. The weight for $\hat{p}_j$, β_{Lj}, is shown explicitly in (6),

$$\hat{w}_j = \beta_{Lj}\hat{p}_j = \lambda_{Lj} \cdot \left[(\sigma_j/\theta_{Tj}) \Big/ \sum_i \lambda_{Li}(\sigma_i/\theta_{Ti})\right] \cdot \hat{p}_j . \qquad (6)$$

The λ_{Li} denote the fraction of the labor force used to produce commodity i. The θ_{Ti} indicate the distributive share of the specific factor in the ith industry. With σ_i expressing the elasticity of substitution between labor and the specific factor in industry i, the expression σ_i/θ_{Ti} is the elasticity of labor's marginal product curve in the ith sector (see [2, pp. 7–8]). Thus the denominator of β_{Lj} is the weighted average over all sectors of the elasticity of labor's marginal product curve in each sector.

In this Ricardo–Viner model any commodity j could be neutral, depending both on factor intensities (as revealed by the λ_{Lj}) and elasticities. A special case leading to unbiasedness would have:

(i) The elasticity of the marginal product of labor schedule for the jth industry, σ_j/θ_{Tj}, is neither greater nor less than the economy-wide average, and

(ii) The jth sector is neither more nor less labor intensive than the economy as a whole. That is, the fraction of the labor force used to produce j, λ_{Lj}, equals the fraction of the national income represented by the value of production in the jth sector, α_j.[5] Assumption (i) implies that β_{Lj} equals λ_{Lj} and assumption (ii) states that λ_{Lj} equals α_j. Thus commodity j is unbiased with respect to j.

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[5] Factor intensity rankings could equivalently be given by distributive shares. Assumption (ii) is the same as having labor's distributive share in the jth sector be equal to labor's distributive share in the economy as a whole.

From these remarks it is clear that commodity j would be biased in favor of labor if j is labor intensive (and (i) still holds). In the Stolper–Samuelson model labor intensity is everything, in fact β_{Lj} exceeds unity if j is labor intensive. (In the Ricardo–Viner model β_{Lj} must remain less than unity.) But elasticities of factor substitution count as well. Even if commodity j is not labor intensive it could still be biased towards labor if labor's marginal product schedule is sufficiently elastic relative to the national average. If it is, the expanding jth sector must absorb relatively much labor from the rest of the economy, thus driving up the wage rate relatively more than if j absorbed little extra labor.

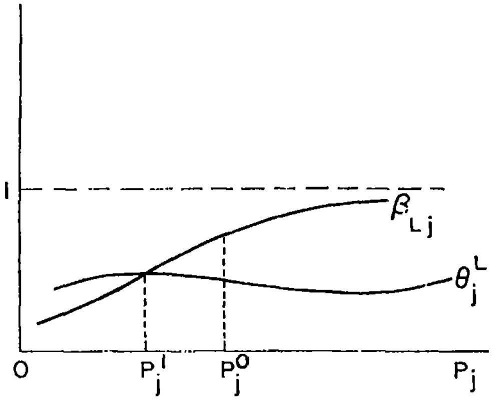

FIG. 2. Suppose commodity j is labor intensive in the Ricardo–Viner model. Autarky is at price p_j^0.

To fix on a specific case where neutrality does not obtain, suppose the jth sector is biased towards labor because it is labor intensive. The β_{Lj} schedule in Fig. 2 has shifted upward from its position in Fig. 1. p_j^0 represents the autarky price. Commodity j is exported for prices higher than p_j^0. If it is exported, any policy that raises p_j benefits labor, as in the neutral case. But suppose commodity j is imported. As Fig. 2 illustrates, an increase in p_j might help rather than harm labor's real position. Much depends on the extent of imports: If the price of commodity j is sufficiently lower than it would be in autarky (i.e., less than p_j^1 as well as p_j^0), a small degree of protection would lower workers' real income. But a sufficiently large tariff (bringing p_j above p_j^1) begins to raise the real wage—"protection begets protection."

This case illustrates that labor is more apt to gain from protecting imports if they are labor intensive. But unlike the Stolper–Samuelson result there are several qualifications. As Fig. 2 illustrates, imports may be so high (production and domestic price so low) that β_{Lj} still falls short of labor's budget share, θ_j^L, so that a small price increase harms workers. Furthermore, a relatively low elasticity of labor's marginal product curve in industry j and/or a higher than average degree of spending by labor on the jth commodity would shift intersection price p_j^1 in Fig. 2 rightward. If this intersection point lies to the right of autarky price p_j^0, protection unambiguously harms labor's real income.

As a final comment on the Ricardo–Viner model, note that labor is the *only* factor capable of leading to unbiased results. All other factors are specific, and the real return to factor i unambiguously rises if the price of the commodity in which it is used rises, and unambiguously falls if, instead, some other commodity price rises. As neoclassical writers were well aware, the ambiguity as to real returns is a problem not for specific factors but for a factor such as labor used in most sectors of the economy.

IV. REDISTRIBUTION OF THE TARIFF PROCEEDS

If price changes are brought about by commercial policy, tariff or tax revenue is raised or lost. What can be said about labor's real income if it shares in these proceeds? Of course much depends upon the fraction of tariff or tax revenue that is rebated to each class of inputs. We restrict ourselves here to the assumption that of any change in government revenues, dT, a fraction $w_L V_L / Y$, is handed back to workers.

Concentrate on the case in which p_j is increased by a tariff on imports of commodity j. First we consider the case in which the tariff rate is initially zero, so that

$$dT = p_j M_j \hat{p}_j , \tag{7}$$

since the world price, $p_j{}^*$, is assumed constant, implying that dp_j equals $p_j{}^* \, dt_j$. Given our assumption about redistribution, the change in tariff receipts going to labor, dT_L, is

$$dT_L = (w_L V_L / Y) p_j M_j \hat{p}_j . \tag{8}$$

In the "neutral" case workers' real income fell in the case of a tariff—the expression shown by (5). Comparison with (8) reveals that the loss of real income is exactly offset by the gain from the redistribution of the tariff proceeds. Net of this redistribution, labor neither loses nor gains by the imposition of a small tariff in the "neutral" case. In this sense labor's fortunes are reflective of those of the entire community: a small economy levying a small tariff from a zero level cannot change its real income.

Pursuing this analogy, suppose there initially does exist a finite tariff on imports of commodity j. Standard tariff theory suggests that the economy as a whole loses by a further increase in the tariff. There are no terms of trade effects for the small country but there is a cut in imports, which reduces real income for the community since x_j is obtained more cheaply abroad than it is worth at home. In terms of tariff revenue,

$$dT = p_j M_j \hat{p}_j + T\hat{M}_j . \tag{9}$$

Since an increase in the tariff lowers imports, the change in labor's share of the proceeds, dT_L, must fall short of the expression shown in (8). In the neutral case labor must lose with a finite tariff even if it gets a "fair" share of the tariff proceeds.

V. CONCLUDING REMARKS

When the price of commodity j rises by 10 %, all other prices constant, does the real wage rise or fall? In general this involves a comparison of the impact on money wages with the deteriorating effect of a price rise on workers' consumption budgets. In search of presumption in this matter we suggested a definition of unbiasedness. Commodity j is considered unbiased or neutral with respect to labor if a rise in p_j raises the money wage by the same relative amount as the average of all factor prices. If labor is also neutral in its taste pattern, devoting the same fraction of its income to commodity j as does the rest of the population, a strong theorem emerges: labor benefits in real terms if and only if commodity j is exported. This unbiased case best points out how the effect of a rise in one commodity price hurts labor because the commodity is consumed and helps labor because it is produced, with the net effect depending upon the balance of consumption and production, i.e., the direction of trade.

The Ricardo–Viner model is a particularly simple example of the neoclassical general equilibrium model. Labor is the only mobile productive factor and the increase of any commodity price leaves in doubt the real wage without details of workers' consumption habits. This position contrasts sharply with the Stolper–Samuelson two-sector result in which labor unambiguously gains or loses depending upon factor proportions alone. The Ricardo–Viner model illustrates how the degree of factor substitutability as well as factor intensity rankings help determine whether a commodity is biased towards or away from labor. The basic theorem for the unbiased case, however, holds for any neoclassical model, including those allowing joint production.

APPENDIX: RECIPROCITY AND A DUAL INTERPRETATION OF UNBIASEDNESS

The text of this paper considered commodity j to be unbiased with respect to labor if and only if a rise in j's price, other prices constant, caused the wage rate to rise, relatively, by the average of all relative factor price changes. This relationship among prices has its dual counterpart among outputs and endowments. Consider the following experiment: Raise the community's endowment of labor, holding all other factor endowments constant, and consider the impact on commodity outputs assuming all commodity prices are kept constant. Some outputs will rise, some may fall (depending upon the details of the model). Let the relative change in output of h be denoted by $\hat{x}_h$ and, as before, let α_h be defined as the share of the national income represented by the value of production of commodity h. Then the *average* of the output changes is $\sum_h \alpha_h \hat{x}_h$, and commodity j could be considered unbiased with respect to labor if $\hat{x}_j$ were to equal this average.

With this alternative definition in hand it remains to establish (i) the value of $\hat{x}_j$ in the neutral case, and (ii) the fact that both definitions of lack of bias are equivalent. Take these in order. The value of the national income, Y, is given either by the value of output, $\sum_h p_h x_h$, or the value of factor incomes, $\sum_i w_i V_i$. (V_i is the endowment of factor i, one of which is labor, L.) Differentiate to obtain

$$\sum_h \alpha_h(\hat{p}_h + \hat{x}_h) = \sum_i \alpha^i(\hat{w}_i + \hat{V}_i). \tag{10}$$

Equation (3) displays the equality among the financial variables, leaving, by subtraction, the relationship between output and endowment changes shown by (11):

$$\sum_h \alpha_h \hat{x}_h = \sum_i \alpha^i \hat{V}_i. \tag{11}$$

If commodity j is neutral with respect to labor by the alternative definition, the left-hand side of (11) equals $\hat{x}_j$ and the right-hand side equals $\alpha^L \hat{V}_L$, since labor is the only expanding factor. Alternatively put, the ratio $\hat{x}_j/\hat{V}_L$ is, if j is neutral with respect to labor, equal to labor's share in the national income.

This term, $\hat{x}_j/\hat{V}_L$ at constant commodity prices, could be called the "Rybczynski" term, just as β_{Lj}, defined by Eq. (1), is the 'Stolper–Samuelson" term. Label the Rybczynski term γ_{Lj}:

$$\gamma_{Lj} \equiv \hat{x}_j/\hat{V}_L.$$

To establish that the two versions of neutrality are equivalent consider Samuelson's *reciprocity theorem* [6, p. 10], whereby

$$\partial w_L/\partial p_j = \partial x_j/\partial V_L . \tag{12}$$

The derivative on the left holds endowments constant and on the right holds commodity prices constant. Put into elasticity form Eq. (12) becomes (13), linking the Rybczynski and Stolper–Samuelson expressions (see also [1]):

$$\alpha^L \beta_{Lj} = \alpha_j \gamma_{Lj} . \tag{13}$$

The first definition of neutrality led to β_{Lj} being equal to commodity j's share of the national income, α_j. If this definition is satisfied, γ_{Lj}, the Rybczynski term, must be equal to labor's share, a result flowing directly from the alternative definition of neutrality.

Note that in the case in which only p_j rises and endowments are fixed, the relative increase in the national income, $\hat{Y}$, is given by $\alpha_j \hat{p}_j$. In the case in which only the labor endowment rises and commodity prices are fixed, $\hat{Y}$ is given by $\alpha^L \hat{V}_L$. Therefore, the lack of bias of commodity j with respect to labor could be stated to require either the relative wage change, $\hat{w}_L$, or the relative output change, $\hat{x}_j$, to equal the proportionate change in the national income when, respectively, either p_j rises or the labor endowment rises. In either case the proportionate change in the national income serves as an average of all factor price or output changes.

Finally, consider the well-known proposition in growth theory that makes use of the alternative definition of neutrality. Suppose the labor force expands and that at initial prices the output of commodity j expands by the same amount, relatively, as the national income, so that commodity j is unbiased with respect to labor. Suppose, furthermore, that demand for j expands also in the same proportion as the national income. Will excess demand or supply be created for commodity j? Excess demand for j is created, in this neutral case, if and only if commodity j is imported. In the neutral case the ultimate direction of change in the price of j depends only upon the direction of trade, so that neutral growth always causes the terms of trade to deteriorate.

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A Unified Model of Consumption, Labor Supply, and Job Search*

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I. INTRODUCTION

In this paper, I unify the two major approaches of the microeconomic foundations literature on aggregate labor force behavior. One approach, exemplified by Mortensen's work [13], assumes the individual is an income maximizer with fixed leisure time and examines the division of his nonleisure time between labor and job search. The second approach, exemplified by Lucas and Rapping's work [11], assumes the individual is a utility maximizer and examines the division of his time between labor and leisure.

Each of these approaches ignores an aspect of the individual's time allocation decision. In the first approach, the fixity of leisure time implies there is no participation decision. The individual always is a participant, so that the labor force is constant. Consequently, the Mortensen approach sheds no light on questions of labor force variability. In the second approach, job search is treated as leisure; this treatment ignores the important insight of the job search literature that search is a productive use of time. It also renders impossible a natural definition of unemployment, so that discussions of unemployment phenomena, such as the Phillips curve, are strained at best.

The two approaches are complementary and should be unified for the sake of completeness alone. However, the need for unification is stronger than merely theoretical nicety. The incompleteness of each of the standard models

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Most of the formal mathematics and analytical detail have been suppressed for the sake of brevity and clarity. They are contained in a longer paper available from the author upon request.

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renders their conclusions suspect. For example, in the Lucas and Rapping model, an increase in current wages reduces leisure if "the" substitution effect is assumed dominant. However, if the individual has search as a third use of time, the response of leisure becomes ambiguous. Labor will rise, but search will fall. When there are three uses of time, the substitution effects on labor and leisure are not the same and must be distinguished. Similar problems exist with the Mortensen approach. Clearly, unification of the two approaches should provide useful insights, as Mortensen [14] and Whipple [24] have suggested.

In this paper, I provide just such a unification. I develop a model in which a utility maximizing individual chooses optimal paths of consumption, labor, search, and leisure. The individual may engage in labor, search, and leisure simultaneously. I discuss the life cycle behavior of the optimal plan and then discuss how the plan changes in response to certain exogenous changes. Most of the earlier microeconomic foundations models are obtained as special cases of the model developed here. A rather surprising finding is that, in the absence of discrete differential costs,[1] the individual never stops searching, always plans to be working in the last part of his life, and thus never retires from the labor force. However, differential costs do exist; an important one in the labor market is that the labor supply is not freely variable. In general, an individual must choose between working some minimum number of hours a week (roughly 40 in the United States) or not working at all. In the last part of the paper, I examine the effect of this institutional arrangement. The major result is that full retirement from the labor force becomes possible.

Because the individual in this model can vary simultaneously all three of his uses of time, it becomes possible to define all three states of labor force participation—employment, unemployment, and nonparticipation—in a natural, mutually consistent way. The application of these definitions to macroeconomic questions is beyond the scope of this paper, but some of the obtainable results are mentioned at the end. A full discussion is deferred to a future paper.

The job search literature is closely related to much of the human capital literature. The model developed herein, with minor changes, can be applied to such human capital problems as the optimal path of schooling. Indeed, many of the human capital models can be obtained as special cases of appropriate variants of the model developed here.[2] These applications, as well as the similarity of my results to some of those obtained in the human capital literature, will be pointed out along the way.

[1] A discrete differential cost is one, like a start-up cost, that jumps to a higher value with the next infinitesimal increment in the activity in question.

[2] Indeed, after the present paper was submitted for publication, Blinder and Weiss [3] published a general human capital model that is much like the model developed here. A comparison of their model with mine is instructive.

II. BEHAVIORAL ASSUMPTIONS

In this section, I discuss the individual's utility function and the asset and wage change constraints the individual faces in trying to maximize lifetime utility.

II. 1. *Utility Function*

Assume the individual derives utility from consumption and leisure. Consumption at any moment t, denoted $C(t)$, is nonnegative and is measured in goods per week. Leisure is all the time not spent in labor and job search. The rate of leisure at any moment t, measured in hours per week, is $168 - L(t) - S(t)$, where 168 is the number of hours in a week and $L(t)$ and $S(t)$ are the rates of labor and search at moment t, measured in hours per week. Labor is time spent working on a job, and job search is time spent looking for a higher paying job than that currently held. The rates of labor and search, and also the sum of these rates, must be between 0 and 168.

The assumption that the individual gets utility from consumption and leisure implies that his utility depends only on C and the sum $L + S$, and not on the sizes of the individual terms L and S. Consequently, utility at time t can be expressed as $U[C(t), L(t) + S(t)]$. I assume that U is concave in consumption and leisure, is additively separable in C and $L + S$, and satisfies the usual Inada conditions; thus

$$
\begin{gathered}
U_1 > 0, \qquad U_{11} < 0, \\
U_2 < 0, \qquad U_{22} < 0, \\
U_{12} = U_{21} = 0, \\
\lim_{C \to 0} U_1 = +\infty, \qquad \lim_{C \to \infty} U_1 = 0, \\
\lim_{L+S \to 0} U_2 = 0, \qquad \lim_{L+S \to 168} U_2 = -\infty.
\end{gathered}
$$

II.2. *The Budget Constraint*

The individual's lifetime budget constraint is

$$(d/dt)[A(t)/p] = r^*[A(t)/p] + [w(t)/p]\, L(t) - C(t), \tag{1}$$

$$A(T) = A_T, \tag{2}$$

$$A(D) = 0, \tag{3}$$

where

$A(t)$ = expected total assets at time t, measured in current dollars;

r^* = the expected interest rate at time t, assumed for now to be constant;

$w(t)$ the expected nominal hourly wage rate at time t, measured in current dollars;

p the expected price of consumption goods at time t, assumed for now to be constant;

T the present moment;

D the moment of death, assumed known by the individual;

A_T the initial value of assets, known by the individual.

For simplicity, I assume that all assets have variable interest rates.

II.3. *Wage Improvement*

The individual has some control over the future value of his wage through job search.

Assume that:

(a) There is a distribution of nominal wages over vacancies, denoted F^*.[3] Let f^* be the density function associated with F^*:

$$f^* = dF^*/dw.$$

(b) The individual has a perception, denoted by F, of F^*. Let f be the density function associated with F.

(c) The individual adjusts F to F^* with a lag. I leave the precise method of adjustment unspecified.

(d) The individual's nominal wage will not change unless he searches.

(e) The individual must accept or reject an offer as soon as he receives it.

(f) Similarly, if the individual quits his job, he loses all rights to it; should he later want to return to it, he would have to compete with other applicants for it.

(g) There is no cost to changing jobs.

(h) The only costs of search are foregone leisure and foregone earnings.

At any moment t, the rate at which the individual contacts vacancies is a function of the rate of search at moment t. Let N be this function, so that at any moment the rate of contact is $N[S(t)]$, which has the units vacancies per week. Other possible arguments of N, such as the vacancy and unemployment rates or transportation technology, are suppressed. I assume that N is concave in search:

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[3] I do not explain the existence of such a distribution; see Rothschild [17] for a discussion of several possible explanations. Also, I assume the distribution is sufficiently well behaved to avoid certain perversities that may arise when the individual is ignorant of the true distribution from which he is sampling; see Rothschild [18] for a discussion of this issue.

$$N'(S) > 0, \qquad N''(S) < 0,$$

and that N' has the limit properties

$$\lim_{S\to 0} N'(S) = +\infty, \qquad \lim_{S\to +\infty} N'(S) = 0.$$

Notice that under these assumptions $N'(S)$ is bounded away from zero for $S \leq 168$.

Suppose an individual earning wage $w(t)$ decides to look into a vacancy chosen at random. His estimate of the probability that he fails to improve his wage is $F[w(t)]$. Suppose the individual searches for a length of time Δt, and suppose $N[S(t)]$ is constant over Δt. The total number of vacancies the individual can contact is $N[S(t)]\,\Delta t$. His estimate of the probability that he will fail to improve his wage during the entire period Δt is

$$(F[w(t)])^{N[S(t)]\Delta t}.$$

The density function associated with $(F[w(t)])^{N[S(t)]\Delta t}$ is

$$(d/dw)(F[w(t)])^{(N[S(t)])\Delta t},$$

which I denote by $f_{N,\Delta t}[w(t)]$. At time t, the expected improvement in the wage over the period Δt, denoted $E[\Delta w(t)]$, is

$$E[\Delta w(t)] = \int_{w(t)}^{\bar{w}} [\omega - w(t)]\, f_{N,\Delta t}(\omega)\, d\omega,$$

where $\bar{w}$ is the maximum wage attainable. This function can be manipulated into the more manageable form

$$E[\Delta w(t)] = \int_{w(t)}^{\bar{w}} (1 - [F(\omega)]^{N[S(t)]\Delta t})\, d\omega.$$

The expected *rate* of change of $w(t)$ at time t, denoted $E[\dot{w}(t)]$, is obtained by taking the appropriate limit as Δt goes to zero:

$$E[\dot{w}(t)] = -N[S(t)] \int_{w(t)}^{\bar{w}} \ln F(\omega)\, d\omega \geq 0.$$

I define

$$E[\dot{w}(t)] \equiv g[w(t), S(t)]. \tag{4}$$

The function g has the derivatives

$$\begin{aligned} g_w &\lessgtr 0, & g_{ww} &\geqslant 0, \\ g_S &\geqslant 0, & g_{SS} &\lessgtr 0, \\ g_{Sw} &\lessgtr 0. \end{aligned}$$

Relation (4) is the individual's best estimate of the future rate of nominal wage improvement. I assume that the individual treats it as the rate that actually will occur, and write[4]

$$\dot{w}(t) = g[w(t), S(t)].$$

The rate of change of *real* wages is simply

$$(d/dt)[w(t)/p] = g[w(t), S(t)]/p, \tag{5}$$

which is the needed function relating wage improvement to job search.

III. The Optimal Solution

I now use the results of Section II to solve the individual's maximization problem.

III.1. *Utility Maximization*

The individual's utility over his remaining lifetime is[5]

$$\int_T^D U[C(t), L(t) + S(t)]\, dt. \tag{6}$$

The individual's problem is to maximize (6) subject to his budget constraint—given by (1), (2), and (3)—and his real wage improvement constraint—given by (5) together with the initial condition

$$w(T) = w_T, \tag{7}$$

where w_T is known by the individual.

This formulation, which maximizes utility as a function of expected wages, eliminates the stochastic elements of the problem and ignores risk. It would be better to treat these problems by maximizing expected utility; however, to do so would require solving a difficult stochastic control problem. I have simplified matters by solving a deterministic approximation instead. I discuss some of the stochastic aspects of the problem with an indirect method presently.[6]

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[4] This assumption implies certainty equivalence and ignores behavior toward risk. I will remark on this simplification presently.

[5] Insertion into (6) of the usual discount factor $e^{-\delta t}$ would not alter the results substantially. However, when the time horizon is known and finite, the discount rate neither is needed to guarantee existence of the integral nor has the natural interpretation that arises when the time horizon is infinite.

As long as A_T is not too negative, as I will assume, a solution or optimal control exists for this problem. The optimal control can be characterized with Pontryagin's maximum principle. The Hamiltonian is

$$H = U[C(t), L(t) + S(t)] + \psi(t)\left[r^* \frac{A(t)}{p} + \frac{w(t)}{p} L(t) - C(t)\right] + \lambda(t) \frac{g[w(t), S(t)]}{p}.$$

If a control is optimal, then

$$(d/dt)[A(t)/p] = \partial H/\partial\psi = r^*[A(t)/p] + [w(t)/p]\, L(t) - C(t), \tag{8}$$

$$(d/dt)[w(t)/p] = \partial H/\partial\lambda = g[w(t), S(t)]/p, \tag{9}$$

$$(d/dt)\, \psi(t) = -\partial H/\partial(A/p) = -r^*\psi(t), \tag{10}$$

$$(d/dt)\, \lambda(t) = -\partial H/\partial(w/p) = -\psi(t)\, L(t) - \lambda(t)\, g_w[w(t), S(t)]. \tag{11}$$

Equations (8) and (9) are simply restatements of (1) and (5). The adjoint variables ψ and λ are the marginal values of assets and wages, respectively, and Eqs. (10) and (11) describe the motion of these variables over time. An optimal control also must satisfy initial conditions (2), (3), and (7); furthermore, by the transversality condition, it must satisfy

$$\lambda(D) = 0. \tag{12}$$

Finally, an optimal control must satisfy the marginal conditions

$$\partial H/\partial C = U_C - \psi = 0, \tag{13}$$

$$\partial H/\partial L = U_L + \psi(w/p) \leqslant 0, \tag{14}$$

$$\partial H/\partial S = U_S + \lambda(g_S/p) = 0. \tag{15}$$

When equality holds in (14), relations (13)–(15) simply express the usual equalities between marginal benefits and marginal costs. However, equality need not hold in (14), as I discuss momentarily.

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[6] For discussions of some of the risk considerations of the time allocation problem, see Whipple [24], Danforth [4], and Levhari and Weiss [10].

Conditions (8)–(15) are necessary for optimal control but may not be sufficient. A standard theorem states that if the Hamiltonian is concave in the state variables, then the necessary conditions are also sufficient. In the problem at hand, the Hamiltonian is convex in w, so the standard theorem does not apply. Fortunately, the concavity of the utility function U in consumption and leisure and the lack of dependence of the objective function (6) on the state variables A and w guarantee that a control for which (2), (3), and (7)–(15) hold is at least locally optimal.[7]

Before discussing the behavioral characteristics of the optimal solution, it is worth pointing out how general the above model is. Most of the earlier micro foundations models are contained as special cases of it. For example, if the general utility maximization of (6) is replaced by a special case (income maximization) and if leisure is fixed, then one obtains the Siven [21] model, which is a continuous-time version of the models of Stigler [22], McCall [12], Mortensen [13, 14], and Gronau [7]. Similarly, if search is set equal to zero and wage growth assumed exogenous, one obtains the Heckman [8] model, which is a continuous-time version of the models of Friedman [5], Lucas and Rapping [11], and Almonacid [1]. Furthermore, solving the model by stochastic rather than deterministic control would provide answers to the questions of risk considered by Whipple [24], Danforth [4], and Levhari and Weiss [10], only in a much more general framework than theirs. The only micro model fundamentally different from the above model is that of Salop [19], in which the job seeker engages in systematic rather than random search.

With minor changes, the above model also encompasses several of the human capital models. In particular, if one replaces the wage rate by the stock of human capital and search by schooling, then one obtains the Blinder and Weiss [3] model, which is a variable-leisure version of Ben-Porath's [2] fixed-leisure model. Finally, if one modifies the wage growth equation (5) to allow wages to grow with the accumulation of work experience, one obtains a generalization of the Weiss [23] model, which is a utility maximizing version of Rosen's [16] income maximizing model.

III.2. *The Optimal Solution*

The Inada conditions imply that consumption and leisure always will be positive. Consumption continuously rises such that the marginal utility of consumption falls at a rate equal to r. This is the same result obtained by Ramsey [15].

Because $U_L = U_S = U_2$, equality in both (14) and (15) would imply that

$$g_S = w\psi/\lambda. \tag{16}$$

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[7] See Lee and Markus [9] for details.

However, the left side of (16) is bounded away from zero, so equality cannot be attained for small values of w (in particular, for w equal to zero). In such cases, L is set equal to zero and S is positive. Thus in the early part of his life, when w is very low, the individual specializes in search (but still takes some leisure). Such specialization also emerges from the earlier search models, such as McCall [12] and Mortensen [13].[8] The major difference here is that the amount of time devoted to search is variable rather than constant. Search may increase or decrease, depending on the properties of the function U and g and on the values of A_T, p, and r. The possibility that search increases as w grows is reminiscent of Ben-Porath's [2] finding that the individual may increase his rate of investment in human capital as human capital grows.

When the individual is not working, (14) holds as an inequality. Relations (13) and (15) hold as equalities; totally differentiating then yields the implicit functions

$$C = C(\underset{(-)}{\psi}), \tag{17}$$

$$S = S(\underset{(+)}{\lambda}, \underset{(-)}{w}), \tag{18}$$

where the sign under a variable is the sign of the partial derivative with respect to that variable. Small changes in ψ, λ, and w have no effect on L, so its partial derivatives are all zero.

A positive value of S implies growth of w. As w grows, equality tends to come about in (16); but until it does, L remains equal to zero. This raises a complication needing explanation. It is clearly sensible to talk about the individual's wage growing when he is working. However, when the individual is searching but not working, he earns no wage. What sense, then, is there in talking about wage growth? Consider an individual at initial time T formulating his optimal plan. Suppose he plans to commence work at time T^* and at an expected wage of $w(T^*)$. Consider a time T' earlier than T^*. The wage $w(T')$ is the expected value of the wage the individual could earn if he followed the same pattern of search but commenced work at time T' instead of T^*. Clearly, $w(T^*)$ exceeds $w(T')$. It is in this sense that the wage grows when the individual is searching but not working.[9]

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[8] Similarly, Ben-Porath [2] shows that there will be an initial period when the individual specializes in accumulating human capital.

[9] If at time T' the individual were unexpectedly ordered by divine decree to commence work at once, the wage he could expect to earn by starting instantly at T' is zero, not $w(T')$. The wage $w(T')$ is the wage that the individual could have expected *at time* T to earn at time T' if he chose to commence work at time T'. However, if everything goes according to plan, when time T' arrives the individual will have passed up all his offers because they

At the moment T^* that (16) begins to hold as an equality, so does (14) and L becomes positive. The time T^* is the optimal expected time for the individual to commence work. The wage $w(T^*)$ corresponds to the "optimal stopping" wage of much of the search literature. Relations (13)–(15) now all hold as equalities and yield the implicit functions

$$C = C(\underset{(-)}{\psi}), \tag{19}$$

$$L = L(\underset{(+)}{\psi}, \underset{(-)}{\lambda}, \underset{(+)}{w}), \tag{20}$$

$$S = S(\underset{(-)}{\psi}, \underset{(+)}{\lambda}, \underset{(-)}{w}), \tag{21}$$

$$L + S = (L + S)(\underset{(+)}{\psi}, \underset{(+)}{w}). \tag{22}$$

It is interesting that λ has equal but opposite effects on L and S, thus having no effect on total nonleisure time $L + S$. Also, the path of consumption is independent of the path of wages. The reason is the separability of U in consumption and leisure. If consumption and leisure had been assumed to be complements, then C and w would be negatively related, reflecting the fact that a higher wage induces a reduction in leisure and therefore makes consumption less desirable. The opposite conclusion would hold if consumption and leisure were substitutes. Heckman [8] obtains the same relation between C and w. As I show in a subsequent section, C does depend on the *initial* value of w and consequently does respond to an *unplanned* change in w.

Once time T^* is reached, the interrelationships of L, S, and $L + S$ become complicated. There will be an interval of time $[T^*, T_1]$ during which L grows; during this time, the substitution effect on L of the growth of w dominates the income effect. If the substitution effect remains dominant and L always grows, then T_1 equals D. If the income effect eventually dominates and L falls, T_1 is less than D; however, L never reaches zero.[10] Consequently,

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will not have offered a high enough wage. So at time T', the individual will command a wage of zero.

These remarks imply an inconsistency in the individual's behavior. Suppose that during the interval $[T, T^*]$ the individual plans to consume at a rate just equal to the rate at which he earns interest on his assets so that A is constant throughout $[T, T^*]$. Then, if everything goes according to plan, the only difference between the individual's state at time T and at time T' is that he is slightly older at T' than at T. Yet at T', the individual expects to wait for a period of length $T^* - T'$ to obtain the wage $w(T^*)$, whereas at time T he expected to wait for a longer period of length $T^* - T$ to obtain the same wage. This inconsistency can be eliminated only by solving the individual's problem with stochastic control.

[10] Because $w(D) > 0$ and $\psi(D) > 0$, Eq. (14) implies $L(D) > 0$.

once L becomes positive, it remains so; and the individual never plans to quit working in order to search. However, as I show in the next section, the individual may quit to search in response to exogenous shocks.

There is an interval $[T^*, T_2]$ during which the behavior of S is ambiguous, as it is before T^*. However, T_2 is less than D; after T_2, S falls, reaching zero exactly at time D.[11] Thus the income effect of w on S eventually dominates the substitution effect unambiguously. If T_1 is less than D, then T_2 may exceed T_1.

Finally, there is an interval $[T^*, T_3]$ during which the behavior of $L + S$ is ambiguous. The relation between T_3 and T_1 and T_2 is in general ambiguous; however, T_3 is unambiguously less than D. After T_3, $L + S$ falls so that leisure rises in the later part of the individual's life. Thus the income effect of w on leisure eventually dominates the substitution effect. This dominance occurs even if L is growing, which means that, even though the substitution effect of w on L dominates the income effect on L (because L is growing), the income effect on leisure dominates the substitution effect. Clearly, it is important to distinguish between the income and substitution effects *on labor* and *on leisure*; they are not the same when a third use of time is present.

In summary, consumption rises continuously throughout the individual's life. In the early part of his life, the individual engages in search but not labor. Once labor becomes positive, it grows for a while; it may continue to grow throughout the lifetime or it may fall eventually. If the substitution effect always dominates, then L always grows. Once positive, labor never falls to zero, being positive even at the moment of death. In the early part of the lifetime, the behavior of search is ambiguous, but eventually search falls, reaching zero exactly at the moment of death D. Leisure behaves ambiguously in the early part of the lifetime but increases in the later part.

Two regions have been defined: (A) Interior: Relations (13), (14), and (15) all hold as equalities, and C, L, and S are all positive. (B) $L = 0$ corner: Relation (15) is a strict inequality and $L = 0$; C and S are positive.

There is no region in which S equals zero, so that the individual always searches. Furthermore, as already noted, the individual never stops working once he has started; in particular, he never retires. These results arise from the absence of differential costs in the model. In Section IV, I discuss the impact of a particular differential cost.

III.3. *Response to Exogenous Shocks*

I now analyze how the optimal plan responds to exogenous shocks. The individual can be thought of as making a new plan at every instant. If at time $t + dt$ all variables have changed as anticipated at time t, then the plan chosen at $t + dt$ coincides on the interval $[t + dt, D]$ with that chosen at time t; otherwise, the new plan differs from the old one. It is the nature of this difference that I study in this section. The analysis provides some insight into the stochastic aspects of the problem.

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[11] $S(D) = 0$ follows from (12) and (15).

An exogenous shock may have different effects in the early and later periods of the individual's life. For example, an increase in the interest rate may cause the individual to increase the labor supply at first but reduce it later. I confine my discussion to the early period and thus to initial effects.

Behavior in region (B) is essentially like that in the interior, except that small shocks may be too small to move the individual out of the corner and thus may have no effect on L. Consequently, corner behavior for the most part is not discussed.

1. *Initial assets* $A(T)$. Suppose that at time T the individual is in the interior and that $A(T)$ unexpectedly increases. Additive separability of the utility function implies that consumption and leisure are normal goods; therefore $C(T)$ increases and $L(T) + S(T)$ decreases.

Intuition suggests that both $L(T)$ and $S(T)$ will fall in response to an increase in $A(T)$, but this does not necessarily happen.[12] However, for simplicity, I assume both L and S fall when $A(T)$ rises.

2. *Rate of interest* r^*. An increase in r^* has a substitution effect tending to decrease $C(T)$ and increase both $L(T)$ and $S(T)$. If the individual's assets are negative, this substitution is strengthened by the income effect. If assets are positive, the income effect tends to offset the substitution effect, leading to ambiguity in the response of $C(T)$, $L(T)$, $S(T)$, and $L(T) + S(T)$.

3. *The function* N. I consider only changes in $N[S(t)]$ that change N and N' in the same direction; other possibilities are mathematically troublesome.

Increases in $N[S(t)]$ and $N'[S(t)]$ raise $g[w(t), S(t)]$ and $g_S[w(t), S(T)]$, thus increasing the returns to search. Clearly, there is a substitution effect tending to increase consumption. Because of the additive separability of the utility function, the income effect also tends to increase consumption. Therefore, $C(T)$ rises in response to the increase in the returns to search. The rise in $C(T)$ implies a fall in $\psi(T)$, which implies by Eq. (22) a fall in $L(T) + S(T)$. Therefore the income effect on leisure of an increase in the returns to search dominates the substitution effect, and leisure rises. Labor unambiguously falls because both the income and substitution effects on labor of the increase in the returns to search tend to reduce labor. The response of search is ambiguous because the income and substitution effects on search oppose each other.

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[12] The increase in $C(T)$ implies that $\psi(T)$ falls, by Eq. (13). In general, the change in $\lambda(T)$ is ambiguous, but it seems safe enough to assume that $\lambda(t)$ falls. If $S(T)$ is to fall in response to an increase in $A(T)$, the right side of (16) must rise. Such a rise requires that the marginal value of wages λ fall by a greater proportion than does the marginal value of assets ψ. Furthermore, if $L(T)$ is to fall as well, then the fall in λ must satisfy $\{W[1 + (\lambda g_{SS}/pU_{SS})]/g_S\}\, d\psi > d\lambda$ for small changes in $A(T)$. In general, there seems to be no reason for the relative changes in ψ and λ to remain within such restrictions.

4. *The wage rate* $w(T)$. Suppose the individual's nominal wage unexpectedly rises with p constant. The income and the substitution effects of the resulting real wage increase both increase consumption, so that $C(T)$ rises and, by Eq. (13), $\psi(T)$ falls. The fall in $\psi(T)$ reduces $L(T) + S(T)$; however, by Eq. (14), the rise in $w(T)$ raises $L(T) + S(T)$. The net effect is ambiguous, and in general one cannot say whether the income or substitution effect on leisure dominates. This conclusion differs from that obtained for an increase in the returns to search. The reason for the difference is that an exogenous increase in w/p has an immediate positive effect on $L + S$, tending to offset the negative effect of the induced fall in ψ. An exogenous increase in N has no such immediate offsetting effects, so that $L(T) + S(T)$ must fall.

Not surprisingly, the response of $L(T)$ to the wage increase is ambiguous, reflecting opposing income and substitution effects. The wage increase raises the individual's absolute wage and also his wage relative to others, thus reducing his real returns to search. There is a double substitution effect reducing search. The absolute wage increase makes labor more attractive relative to search; the reduction in returns to search makes leisure more attractive relative to search. The total effect is negative and therefore reinforces the income effect. Consequently, $S(T)$ falls.

In summary, then, an exogenous increase in $w(T)$ raises $C(T)$, reduces $S(T)$, and has ambiguous impacts on $L(T)$ and $L(T) + S(T)$.

Consider now an equiproportional increase in all nominal wages with p constant. If the individual does not perceive that wages other than his own have changed, he reacts as if only his own wage has changed, as just discussed. However, if the individual does perceive the general nature of the wage change, then he perceives no relative increase in his wage. The distribution F is expanded in such a way that both g/p and g_s/p rise by the same proportion as $w(T)/p$. Consequently, the absolute real returns to both labor and search rise but their ratio is unchanged; there is no change in the relative returns of search and labor. The absolute increase in the returns to search makes leisure less attractive relative to search; therefore there is a substitution effect that tends to increase search, opposing the income effect. The change in $S(T)$ is ambiguous in this case. The responses of the other variables are the same as when only the individual's wage changed.[13]

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[13] Note that if the individual does not perceive that wages other than his own have risen, he will not reduce $L(T)$ to zero and move to region (B), because to do so would make it impossible to capture the benefits of the increase in $w(T)$. However, if the individual does perceive that all wages have risen equiproportionally, he may set $L(T)$ equal to zero because the increased returns to search may induce him to devote all nonleisure time to search.

Note that if the individual is unemployed i.e., in region (B) when the equiproportional wage increase occurs and if he does not perceive the increase, he is completely unaffected by the exogenous change. If he does perceive the increase, then he also perceives an increase in his returns to search. In response, he may increase or decrease $S(T)$. Whichever he does, it seems most likely that he will reduce the expected duration of his unemployment, as I discuss in a subsequent section. Note that he does not become employed immediately at time t because his own wage $w(T)$ is still zero.

5. *The price level p.* Suppose p rises once and for all. The individual's real assets, real wage, and real returns to search decrease equiproportionally. The result is the same as if there had been an equiproportional decrease in nominal assets and wages.

To obtain more interesting results, suppose there is an equiproportional increase in all nominal wages and prices. Real assets fall, and real wages are unchanged. If the individual perceives the equiproportional increase in nominal wages, the perceived real returns to search are unchanged, and the individual experiences only the asset effect and reacts accordingly. If the individual does not perceive the equiproportional increase in nominal wages, then he perceives a decrease in his real returns to search. This perceived decrease causes a decrease in $C(T)$, an increase in $L(T)$ and in $L(T) + S(T)$, and ambiguous change in $S(T)$.

6. *Expected inflation.* Consider now trends in wages and prices. Suppose the rate of wage inflation always equals the rate of price inflation. Suppose the individual forms a perception, π, of this rate. The real interest rate is $r^* = r - \pi$, where r is the nominal rate.

If r adjusts to changes in π so as to keep r^* constant, it is easily seen that changes in π have no effect on the individual's behavior. If r fails to adjust fully to changes in π (because of well-known liquidity effects) such that r^* is inversely related to π, then changes in π cause changes in r^* and the individual reacts as described above.[14]

7. *Summary of results.* The following equations summarize the foregoing results:

$$C(T) = C^E(\underset{+}{A/p}, \underset{?(+,-)}{r - \pi}, \underset{+}{N}, \underset{+}{w/p}); \qquad (23)$$

$$L(T) = L^E(\underset{-}{A/p}, \underset{?(-,+)}{r - \pi}, \underset{-}{N}, \underset{?(-,+)}{w/p}); \qquad (24)$$

$$S(T) = S^E(\underset{-}{A/p}, \underset{?(-,+)}{r - \pi}, \underset{?(-,+)}{N}, \underset{-,\,?(-,+)}{w/p}); \qquad (25)$$

$$L(T) + S(T) = (L + S)^E(\underset{-}{A/p}, \underset{?(-,+)}{r - \pi}, \underset{-}{N}, \underset{?(-,+)}{w/p}). \qquad (26)$$

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[14] When inflation is present, (5) becomes $(d/dt)[w(t)/p] = \{g[w(t), S(t)]/p\} - (w/p)\pi$.

The superscript E is a reminder that these results pertain to exogenous changes in the variables concerned. The variable N stands for all parametric changes that change N and N' in the same direction for all values of S. The variable w/p stands for equiproportional changes in all real wages. The sign under a variable is the sign of the partial derivative with respect to that variable. Signs inside parentheses are the results for dominance of the income and substitution effects, respectively. There are two signs under w/p that are outside parentheses in (25). The first of these is the result when the individual does not perceive that wages other than his own have changed (and is thus also the result when only the individual's wage changes); the second is the result when the individual does perceive this change.

Note finally that all results are for individuals in the interior. For individuals in the $L = 0$ corner, the results in Eq. (24) are only tendencies which may not be realized if the exogenous changes are small.

III.6. *Some Additional Implications for Behavior*

Anything which changes ψ, the marginal value of assets, changes the acceptance wage in the opposite direction. For example, an unexpected increase in $A(T)$ reduces $\psi(t)$ for all $t \geqslant T$. The lower $\psi(T)$ means that the wage required to bring relation (14) into equality is higher than before. On average, then, the wealthier people are, the higher will be their acceptance wage. Similarly, the better the wage distribution that individuals face, the higher will be their acceptance wage because of the higher returns to search. Note in particular that the acceptance wage can be expected to fall as the duration of unemployment increases. As people remain unemployed, they presumably become more pessimistic about the N and F functions and revise "down" their perceptions of them. These downward revisions imply increases in ψ because individuals, perceiving themselves worse off, reduce their consumption. The increases in ψ mean that the acceptance wage has fallen.

The duration of unemployment (the length of stay in region (B)) is positively related to assets. If $A(T)$ unexpectedly increases, an individual in region (B) responds by reducing $\psi(T)$ and $S(T)$. The lower $\psi(T)$ increases the acceptance wage, and the lower $S(T)$ increases the time required to attain any wage. Together, the two changes clearly imply an increase in the duration of unemployment. Voluntary unemployment is a luxury.

The relation between the duration of unemployment and the returns to search depends on how the returns to search are changed. An equiproportional increase in all nominal wages that is perceived by the individual is likely to bring about a decrease in the duration of unemployment. Because the increased return to search arises from an increased attractiveness of labor, most individuals would attempt to become employed sooner. Such reasoning does not apply to an increase in the returns to search that arises from increases in $N(S)$ and $N'(S)$, and there seems to be no reason to assume anything about the change in the duration of unemployment in such a case. This latter result agrees with the implication of the McCall–Mortensen–Siven model [12, 14, 21].

Finally, although the individual never plans to quit work in order to search, he may do so in response to exogenous shocks. An unexpected drop in $A(T)$ or $w(T)$, for example, could have such a result.

IV. Restriction on Hours Worked

So far, I have assumed that the individual can work any number of hours between 0 and 168 that he chooses. A consequence of this assumption is that the individual never sets both L and S equal to zero at once and therefore never is a labor force nonparticipant. In particular, the individual never retires from the labor force. In reality, however, most jobs require that an individual work some minimum number of hours. For convenience, I take this minimum to be uniform at 40 hours because that seems to be about average. However, the analysis of this section does not depend on this choice; any number will do. It is only the existence and not the value of the minimum that matters.

The restriction on hours worked makes retirement and nonparticipation in the labor force possible. It also leads to natural, precise definitions of employment, unemployment, and nonparticipation that are a significant improvement over those found in the earlier micro foundations literature.

IV.1. *The Individual's Problem Reconsidered*

Assume that the individual cannot work for less than 40 hours a week; he can supply less than 40 hours of labor if he wishes, but no one will hire him. In addition, assume that if an individual holding a job decides to supply less than 40 hours of labor per week, he loses his job and the wage rate he commands becomes zero. For simplicity, continue to assume that the individual can work any amount over 40 hours that he desires.

Under these assumptions, the wage earned depends on the number of hours of labor supplied. Let $w(t)$ be the wage rate associated with a certain job but which will be paid only if the individual works at least 40 hours a week. Then the wage actually earned, denoted w_e, is

$$w_e(t) = \begin{cases} w(t), & \text{if } L(t) \geq 40, \\ 0, & \text{if } L(t) < 40. \end{cases}$$

The model of individual behavior is changed somewhat by the presence of the restriction on hours worked. The Hamiltonian becomes[15]

$$H = U(C, L + S) + \psi[r^*(A/p) - (w_e/p)L - C] + \lambda[g(w, S)/p]$$

and Eqs. (12)–(15) become[16]

$$(d/dt)[A(t)/p] = r^*[A(t)/p] + (w_e/p)\,L(t) - C(t), \tag{8'}$$

$$(d/dt)[w(t)/p] = g[w(t), S(t)]/p, \tag{9'}$$

$$(d/dt)\,\psi(t) = -r^*\psi(t), \tag{10'}$$

$$(d/dt)\,\lambda(t) = \lambda(t)\,L(t) - \lambda(t)\,g_w[w(t), S(t)]. \tag{11'}$$

The original endpoint conditions (2), (3), (7), and (12) still hold. Relations (13)-(15) become

$$U_C = \psi(t), \tag{13'}$$

$$U_L \leq -\psi(t)[w_e(t)/p], \tag{14'}$$

$$U_S = -\lambda(t)\{g_S[w(t), S(t)]/p\}. \tag{15'}$$

Suppose $w(t)$ were large enough that (14′) would be satisfied as an equality if there were no restriction on hours worked, that is, the individual would choose to be in region (A) rather than region (B). The equality relation represents what the individual would like to achieve; it is the relation that would maximize his utility in the absence of the constraint. When the restriction on hours worked is present, however, equality in (14′) cannot always be attained—in particular, whenever the individual would like to work a positive amount less than 40 hours. The individual then must choose between working 40 hours, which is more than the unconstrained optimum, and not working at all, which is less than the unconstrained optimum. In either case, his utility is less than the unconstrained case. Also, (14′) is satisfied as an inequality. If the individual chooses to work 40 hours, the marginal disutility of labor exceeds the marginal benefit, which implies that $U_L < -\psi(w/p)$. If the individual chooses not to work, the right side of (14′) becomes zero but the left side remains negative as long as S is positive, because U_T is identically equal to U_S. Therefore, $U_L < 0$. There are three possibilities:

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[15] The restriction on L and the consequent restriction on w_r mean that the velocity set is no longer convex so that the standard existence theorem does not apply. Consequently, an optimal classical control may not exist. However, another theorem guarantees the existence of an optimal chattering control. Chattering controls contain classical controls as special cases, and, as it turns out in this problem, a chattering control that is not classical apparently cannot be optimal. Consequently, for the present problem an optimal classical control does exist.

[16] The variable w continues to appear in (9′) and (11′) because these equations describe the behavior of the wage offer and of the shadow price of a better wage offer. Equation (8′) describes the behavior of real assets, and the relevant wage there is the one actually earned.

(i) $L \geq 40$. The constraint is not binding, and the individual works 40 hours or more.

(ii) $L = 40$. The constraint is binding, but the individual chooses to work.

(iii) $L = 0$. The constraint is binding, and the individual chooses not to work.

When the constraint is binding, the individual must decide when to switch between states (ii) and (iii), that is between working 40 hours a week and not working at all. The pertinent expression is

$$\psi(t) \gtrless \{U[C(t), S_1(t)] - U[C(t), 40 + S_2(t)]\}/40, \tag{27}$$

where $S_1(t)$ is the amount of search that would be undertaken if $L(t)$ were set equal to zero and $S_2(t)$ is the amount of search if $L(t)$ were set equal to 40 (see the Appendix for the derivation of (27)). If the left side exceeds the right, the individual sets $L(t)$ equal to 40; if the right side exceeds the left, the individual sets $L(t)$ equal to 0; and if the two sides are equal, he is indifferent between having $L(t)$ equal 40 and 0.

Because g_S has no upper bound, (15′) holds as an equality, which implies that search will be positive for all $t \leq D$ except in one case.[17] If the individual is not working and does not plan to work again, then search will be zero for the rest of the individual's life. The intuitive reason is that as long as the individual plans to work at some time in the future, there is a return to search, and search will be positive; but if no future work is planned, there is no point to searching, search will be zero for the rest of individual's life, and the individual retires.[18]

An individual retires at time T' if and only if

$$\max \int_{T'}^{D} U[C_0(t), 0]\,dt \geq \max \int_{T'}^{D} U[C(t), L(t) + S(t)]\,dt,$$

where the maximum on the left side is taken over all paths of consumption C_0

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[17] Thus there is still no state in which $L > 0$, $S = 0$. Existence of such a state requires differential search costs, such as start-up costs.

[18] At the moment retirement begins, λ becomes zero and remains zero to time D. When search is positive, so is λ and (15′) is satisfied as an equality; when search is zero, so is λ and again (15′) is satisfied as an equality.

consistent with endpoint condition (3) when L and S are zero from T' to D and where the maximum on the right side is taken over all paths of C, L, and S consistent with (2), (3), (7), (8′)–(11′), and (12). In the model of Section III, the above inequality could not be satisfied because the individual always found it worthwhile to devote at least a small amount of time to labor and was allowed to do so; consequently, there was no retirement.

The functional expressions for C, L, and S depend on whether the individual is in the interior or in one of the corners:

(A) *Interior.* If the individual works more than 40 hours, the restriction on hours worked is not binding, and (13′)–(15′) are satisfied as equalities. The individual behaves as in the interior of the model without the hours restriction. Equations (19)–(22) apply.

(B) *Type I corner*: $L = 40$. The restriction on hours worked is binding, and the individual chooses to work. Then L equals 40, and S is positive. Small changes in ψ, λ, and w have no effect on L, so the partial derivatives of L all are zero. Relation (14′) can be ignored and the resulting system of (13′) and (15′) gives the implicit functions (17) and (18).

(C) *Type II corner*: $L = 0$, $S > 0$. The restriction on hours worked is binding, and the individual chooses not to work now but plans to work later. Then $S > 0$. As at the Type I corner, L is not affected by small changes in ψ, λ, and w, and Eqs. (17) and (18) again apply.[19]

(D) *Type III corner*: $L = 0$, $S = 0$. The restriction on hours worked is binding, and the individual chooses not to work now or in the future. Then $S = 0$. Neither L nor S is affected by small changes in ψ, λ, or w; and all the partial derivatives of both L and S are zero. Relation (14′) and (15′) can be ignored, and relation (13′) gives (17) as the implicit function for C.

The factors that influence the choice of the optimal plan can be studied exactly as in Section III.5. The results are essentially the same as there. The restriction on hours worked simply adds the usual complications of corner behavior.

The major addition is the possibility of nonparticipation and retirement. The individual may plan to retire late in his life. Also, he may leave the labor force in response to exogenous shocks, such as an increase in $A(T)$.

IV.2. *Employment, Unemployment, and Nonparticipation*

Each of the two major types of models in the micro foundations literature ignores an aspect of individual behavior and consequently is incapable of explaining the corresponding aspects of aggregate behavior. The Mortensen [13] model assumes the individual to be an income maximizer with fixed leisure time and examines his allocation of time between search and labor. The fixity of leisure time implies no participation decision and consequently no variability in the labor force. The Lucas and Rapping [11] model assumes the individual to be a utility maximizer never engaging in job search and examines his allocation of time between labor and leisure. The neglect of search implies no response of labor or leisure to changes in search-related variables, such as the distribution of wages, thus ignoring the insights of the recent job search literature. Because of these deficiencies, neither approach can define in a mutually consistent way the three states of labor force participation: employment, unemployment, and nonparticipation. Finally the Lucas and Rapping approach must relate aggregate unemployment to manhours of labor. However, when individuals engage simultaneously in labor, search, and leisure, it is not clear that one can expect a stable relation between unemployment and the manhours devoted to one particular use of time. It is preferable to analyze unemployment directly in numbers of people rather than through a manhours proxy.

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[19] Region (B), where $L = 0$ and $S > 0$, of the model in Section III is included in the Type II corner. The choice of not working at all can be treated as the extreme case of supplying less than 40 hours of labor.

The model developed in this paper overcomes all these difficulties. By allowing simultaneous labor, search, and leisure, the model permits natural, direct definition of employment, unemployment, and nonparticipation:

(i) Employment. The individual is in either the interior or the Type I corner.

(ii) Unemployment. The individual is in the Type II corner.

(iii) Nonparticipation. The individual is in the Type III corner.

Aggregate employment, unemployment, and nonparticipation are simply the totals of all employed, unemployed, and nonparticipating individuals. There are two great virtues to these aggregate definitions:

(a) All three of the states of labor force participation are defined simultaneously and in a mutually consistent way.

(b) The definitions are not in terms of functions of hours of labor, search, or leisure, but are in terms of whether or not individuals are working (irrespective of how many hours they work) and whether or not individuals are searching (irrespective of how many hours they search).

The application of these definitions and the model on which they are based to aggregate analysis is beyond the scope of this article and will be discussed in a future paper. I give only a brief summary here.

First, even though the response of any given individual to an exogenous change may be ambiguous in some ways because of income and substitution effects, this ambiguity does *not* carry over to the responses of employment, unemployment, and nonparticipation. The responses of employment and nonparticipation are determinate for most exogenous changes of interest. The response of unemployment generally *is* ambiguous, *not* because of income and substitution effects but because unemployment is a "middle" state between employment and nonparticipation. Changes which tend to induce some people to leave unemployment for employment also tend to induce other people to leave nonparticipation for unemployment, leaving the net change in unemployment ambiguous. This result is impossible in either the Mortensen or the Lucas and Rapping framework because, not being able to consider all three states of participation, those frameworks cannot properly treat unemployment as a middle state.

Second, under appropriate assumptions about perceptions, the model can generate a negatively sloped short-run relation between inflation and unemployment. Under most sets of assumptions used in earlier contributions, such as those of Friedman [5] and Moretensen [13], the model's long-run Phillips curve is vertical. However, under the assumptions used by Lucas and Rapping [11], the relation between inflation and unemployment is ambiguous in both the short run and the long run, although the relation between both employment and nonparticipation and inflation is unambiguous in both the short and long run. The results on inflation and unemployment differ from those obtained by Lucas and Rapping.

V. SUMMARY

In this paper, I have constructed a model of individual behavior that unifies the two dominant approaches to the study of the micro foundations of aggregate labor force behavior. Several earlier models of individual behavior, including some human capital models, have been derived as special cases.

The model's major contribution to the study of individual behavior is its analysis of the individual's simultaneous choice of all three uses of time: labor, search, and leisure. This analysis permits extensive examination of the individual's optimal plan, the response of that plan to various exogenous shocks, and the behavior of certain related variables such as the acceptance wage and the duration of unemployment. The model's major omission is analysis of risk. Such analysis would be a most worthwhile extension. The model's major contribution to the study of aggregate behavior is the natural and precise set of definitions of all three states of labor force participation: employment, unemployment, and nonparticipation. Macroeconomic applications are deferred to a future paper.

APPENDIX: DERIVATION OF (27)

At each instant t, the benefit from working at a rate of 40 hours a week is

$$U_c[w(t)/p]\,40,$$

which is the marginal utility of consumption times the real hourly wage rate times the number of hours worked in a week. By (13′), U_C can be replaced by $\psi(t)$ to give

$$\psi(t)[w(t)/p]\,40$$

as the benefit to working 40 hours a week.

The cost at instant t of working 40 hours a week instead of not at all is

$$U[C(t), S_1(t)] - U[C(t), 40 + S_2(t)].$$

$S_1(t)$ is the value of $S(t)$ given that $L(t)$ equals zero, and $S_2(t)$ is the value of $S(t)$ given that $L(t)$ equals 40. Notice that $C(t)$ is determined only by the given value of $\psi(t)$ and does not depend on whether L is set equal to 40 or 0.

As long as $40 + S_2(t)$ exceeds $S_1(t)$, the cost of working 40 hours a week instead of not at all is positive. In fact, $40 + S_2(t)$ always does exceed $S_1(t)$. Suppose the individual starts with $L = 0$ and $S = S_1$ and then raises L to 40. From (15′), the sudden increase in L must mean a drop in S to some new value S_2. Suppose that S were to drop by as much as L rose; then U_S and therefore the left side of (15′) would be unchanged because $L + S$ would be unchanged. However, g_S would be higher because S had decreased. This implies that the right side of (15′) falls and no longer equals the left side. To restore equality, S would have to be raised, implying a rise in $L + S$. Consequently, $40 + S_2 > S_1$.

(The foregoing analysis also implies that $S_1 > S_2$, but there is a case when this inequality does not hold. If the individual moves from region (*ii*), where $L = 0$, and if he simultaneously sets $S = 0$, then $S_2 > S_1$, where $L = 40$, to region (iii), where $S_1 = 0$.)

To decide whether to set $L(t)$ equal to 40 or 0, the individual compares the benefit and cost of having $L(t)$ equal to 40:

$$\psi(t)[w(t)/p]\,40 \gtreqless U[C(t), S_1(t)] - U[C(t), 40 + S_2(t)].$$

This expression can be rewritten as

$$\psi(t)[w(t)/p] \gtreqless \{U[C(t), S_1(t)] - U[C(t), 40 + S_2(t)]\}/40,$$

which is relation (27).

Notice that the right side of relation (27) can be written as

$$-\{U[C(t), 40 + S_2(t)] - U[C(t), S_1(t)]\}/40$$

If the minimum number of hours of labor required, now set at 40, is allowed to approach zero, then $S_2 \to S_1$ and

$$-\{U[C(t), 40 + S_2(t)] - U[C(t), S_1(t)]\}/40 \to -U_L .$$

In the limit, then, relation (27) becomes relation (14).

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Mixed Pricing in Oligopoly*†

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1

A ubiquitous economic reality we encounter every day is that a product is sold for different prices by different sellers. Because the perfect-competition model of our textbooks cannot account for this reality, considerable theoretical effort has gone into explaining it (see [7] for a survey).

Most models, for example, those of Fisher [3], Rothschild [6], Diamond [2], and Winter [9], present the price dispersion as a temporary phenomenon that fades away as the model converges, at least in probability, to a single-price equilibrium. Other models either explain price distribution over time and not across firms, like Green and Majumdar's [4], or lack interaction and competition among sellers, like Rothschild and Yaari's in [7].

This paper offers a game-theoretic hypothesis to account for the phenomenon of price dispersion. In spirit, it is close to the duopoly models with "unstable" prices that have appeared since the creation of Edgeworth's model (see [5, 8] for an account.) It models both sides of the market: consumers are clever shoppers and sellers are engaged in a noncooperative game and must be concerned with their competitors' actions. The motivation is the observation in some markets of erratic price fluctuations and sale policies that suggest some element of deliberate randomization. (A model with slight similarity in spirit to the one presented here was independently developed by Butters [1] at almost the same time.)

The model assumes some segmentation of the market according to location, brand loyalty, or access to information, and a costly movement across segments for consumers. While the language of the locational application is

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used throughout the paper, comments on some of the other applications are given in the last section.

Section 2 presents the problem, which is solved in Section 3. In Section 4, there are issues of uniqueness; and, in Section 5, short comments on stability. Section 6 gives concluding remarks and extensions.

2

Consider the following situation: n sellers of a commodity are scattered in different neighborhoods around a city or in adjacent cities. In each locality, there is a totally inelastic demand for one unit of the good for prices between zero and one. Once a week, say, every seller decides a price and puts an advertisement to that effect in the local newspaper, not knowing the prices being determined and sent to the newspapers by his competitors. When all prices are posted consumers shop for the better bargain, taking into account the fact that there is a transportation cost c between any two locations.

By charging for the good a price higher by more than c than that of the cheapest seller, a seller prices himself out of the market and his customers desert him for a better deal. A price low enough will bring a seller a flood of customers from all n markets. If two or more sellers charge the minimum price, it is assumed that they share the spoils.

For $n = 2$, the payoffs to the players for different combinations of prices (p_1, p_2) are graphed in Fig. 1. The first numbers are the first player's payoffs; and the second numbers, the second's.

In general, the payoff function of the kth seller has the form

$$\pi_k(p_1, \ldots, p_n) = \begin{cases} 0, & \operatorname{Min}(p_1, \ldots, p_n) + c < p_k, \\ p_k, & \operatorname{Min}(p_1, \ldots, p_n) < p_k \leq \operatorname{Min}(p_1, \ldots, p_n) + c, \\ \left(1 + \dfrac{j}{i+1}\right) p_k, & p_k = \operatorname{Min}(p_1, \ldots, p_n), \\ & |\{m \neq k : p_m = \operatorname{Min}(p_1, \ldots, p_n)\}| = i, \\ & \text{and } |\{m : p_m > p_k + c\}| = j. \end{cases}$$

This noncooperative n-person game has no Nash solution in pure strategies. At every point $(p_1, \ldots, p_n)$, a seller with zero payoff, because his price is too high, would surely like to reduce his price; if there is no seller with zero payoff, everyone sells only to his own market and a seller can gain by increasing his price as long as the difference between his price and $\operatorname{Min}(p_1, \ldots, p_n)$ is not more than c. At $(1, \ldots, 1)$, each one has an incentive to cut his price to slightly below $1 - c$ and multiply his clientele.

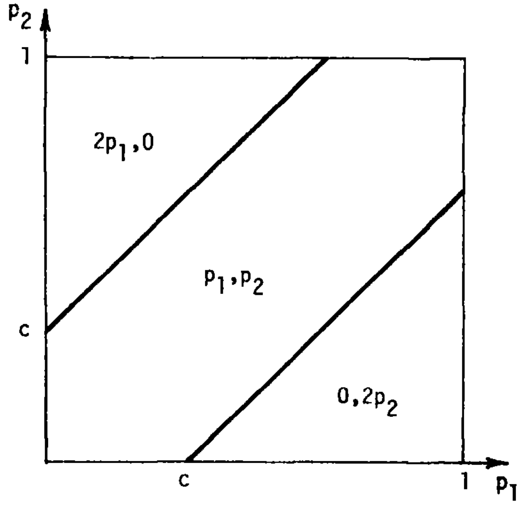

FIGURE 1

A seller can guarantee himself only the low payoff c by charging price c. It will be proved that the sellers can expect better by using mixed strategies and gain a value higher than c. Interestingly, the value approaches c as n gets larger. Zero production costs are assumed here; constant marginal cost will give the same result.

3

CLAIM. The following cumulative distribution function, when employed by all players, is a solution of the game; furthermore, it is the only such symmetrical solution:

$$
\begin{aligned}
F(p) &= 0, && p < V - c,\\
&= 1 - [V/(p+c)]^{1/(n-1)}, && V - c \leq p < \bar{p} - c,\\
&= 1 - (V/\bar{p})^{1/(n-1)}, && \bar{p} - c \leq p < V,\\
&= 1 - \frac{V/(p-c) - 1}{(n-1)(V/p)^{(n-2)/(n-1)}}, && V \leq p < \bar{p},\\
&= 1, && \bar{p} \leq p,
\end{aligned}
$$

where V is the value of the game depending on c; $V = ac$ for $c \leq 1/(a+1)$ and V is the solution of

$$\left[\frac{c}{(n-1)(V-c)}\right]^{n-1} = V, \qquad \text{for} \quad \frac{1}{a+1} \leq c < \frac{n-1}{n};$$

a is the solution of

$$\frac{a}{a-1} = \left[\frac{1}{(n-1)(a-1)}\right]^{n-1};$$

$\bar{p}$ is the highest charged price;

$$\begin{aligned} \bar{p} &= V + c, && \text{for} \quad c < 1/(a+1), \\ &= 1, && \text{for} \quad 1/(a+1) \leq c. \end{aligned}$$

For $c \geq (n-1)/n$, F degenerates to a pure strategy at $p = 1$.

Proof. The proof is by construction. The solution is found by examining the conditions it must satisfy. These conditions are sufficient to specify the solution. Assumptions made in the course of the construction are proved to be necessary.

Suppose each player employs as his mixed strategy a cumulative distribution function $F(p)$. The expected payoff to the first player, say, when he charges price $p_1 = p$ and the other $n-1$ players use F, assuming F is continuous at p and at $p + c$, is

$$\begin{aligned} &E\pi_1(p, p_2, \ldots, p_n) \\ &\quad = \int_0^1 \cdots \int_0^1 \pi_1(p, p_2, \ldots, p_n)\, dF_2(p_2)\, dF_3(p_3) \cdots dF_n(p_n) \\ &\quad = p\left\{\sum_{i=1}^{n-1} \binom{n-1}{i} [F(p) - F(p-c)]^i\, [1 - F(p)]^{n-1-i}\right. \\ &\qquad + [F(p+c) - F(p)]^{n-1} \\ &\qquad \left. + \sum_{i=1}^{n-1} (i+1) \binom{n-1}{i} [1 - F(p+c)]^i\, [F(p+c) - F(p)]^{n-1-i}\right\} \\ &\quad = p\{[1 - F(p-c)]^{n-1} - [1 - F(p)]^{n-1} + [F(p+c) - F(p)]^{n-1} \\ &\qquad + (n-1)[1 - F(p+c)][1 - F(p)]^{n-2} \\ &\qquad + [1 - F(p)]^{n-1} - [F(p+c) - F(p)]^{n-1}\} \\ &\quad = p\{[1 - F(p-c)]^{n-1} + (n-1)[1 - F(p+c)][1 - F(p)]^{n-2}\}. \end{aligned}$$

If F is a solution, it must satisfy for almost every p in its support $E\pi_1(p, p_2, \ldots, p_n) = V$, where V is the value of the game for the first player. Thus we get the difference equation that F must satisfy at almost every point p such that F is continuous at p and at $p - c$:

$$[1 - F(p-c)]^{n-1} + (n-1)[1 - F(p+c)][1 - F(p)]^{n-2} = V/p. \quad (1)$$

We shall find such F whose support is contained in an interval not larger than $2c$. In Section 4 we prove that there are no solutions with wider range. Let $\underline{p}$ be the lowest price charged and $\bar{p}$ the highest. Then we assume

$$0 \leqslant \underline{p} < \bar{p} - c < \underline{p} + c < \bar{p} \leqslant 1. \tag{2}$$

For $\underline{p} + c \leqslant p < \bar{p}$, $F(p + c) = 1$, so (1) becomes $[1 - F(p - c)]^{n-1} = V/p$ from which we get

$$F(p - c) = 1 - (V/p)^{1/(n-1)}, \qquad \underline{p} + c \leqslant p < \bar{p},$$

or (3)

$$F(p) = 1 - (V/(p + c))^{1/(n-1)}, \qquad \underline{p} \leqslant p < \bar{p} - c.$$

For $\underline{p} \leqslant p < \bar{p} - c$, $F(p - c) = 0$, so (1) becomes

$$1 + (n - 1)[1 - F(p + c)][1 - F(p)]^{n-2} = V/p;$$

substituting (3) in it we get

$$1 + (n - 1)[1 - F(p + c)](V/(p + c))^{(n-2)/(n-1)} = V/p$$

or (4)

$$F(p) = 1 - \frac{V/(p - c) - 1}{(n - 1)(V/p)^{(n-2)/(n-1)}}, \qquad \underline{p} + c < p < \bar{p}.$$

For $\bar{p} - c \leqslant p < \underline{p} + c$, F will be flat since this range is all in the middle range of the payoff function, where one sells only to his own segment of the market, and the payoff can be increased by charging more. Together, we get the c.d. function

$$\begin{aligned}
F(p) &= 0, && p < \underline{p},\\
&= 1 - (V/(p + c))^{1/(n-1)}, && \underline{p} \leqslant p \leqslant \bar{p} - c,\\
&= 1 - (V/\bar{p})^{1/(n-1)}, && \bar{p} - c \leqslant p < \underline{p} + c,\\
&= 1 - \frac{(V/(p - c)) - 1}{(n - 1)(V/p)^{(n-2)/(n-1)}}, && \underline{p} + c < p < \bar{p},\\
&= 1, && \bar{p} \leqslant p.
\end{aligned}$$

This defines F for all points p such that F is continuous at p and $p - c$. We shall show that a discontinuity is possible only at $p = \bar{p}$ if F is to be a solution.

First, we observe that only three points are candidates for discontinuity: $\underline{p}$, $\underline{p} + c$, $\bar{p}$. This is because the set of all discontinuity points of a c.d.f. is denumerable so that in every neighborhood of a discontinuity point p_0 there are points p, where F is continuous and must follow (3) if $\underline{p} < p < \underline{p} + c$, must be flat for $\bar{p} - c < p < \underline{p} + c$, and must follow (4) if $\underline{p} + c < p < \bar{p}$. That precludes a jump at p_0.

Next, suppose $F(\underline{p}) = a > 0$. The payoff to the first player when charging $\underline{p}$ is

$$
\begin{aligned}
E\pi_1(\underline{p}, p_2, \ldots, p_n) &= \underline{p}\left\{\sum_{j=0}^{n-1}\binom{n-1}{j} a^j \sum_{i=0}^{n-1-j}\left(\frac{i}{j+1}+1\right)\binom{n-1-j}{i}\right.\\
&\qquad \left.\cdot\, [1 - F(\underline{p}+c)]^i\,[F(\underline{p}+c) - a]^{n-1-j-i}\right\}\\
&= \underline{p}\left\{\sum_{j=0}^{n-1}\binom{n-1}{j} a^j \left(\frac{n-1-j}{j+1}[1 - F(\underline{p}+c)]\right.\right.\\
&\qquad \left.\left.\cdot\, (1-a)^{n-2-j} + (1-a)^{n-1-j}\right)\right\}\\
&= \underline{p}\left\{\frac{1-F(\underline{p}+c)}{a}\sum_{j=0}^{n-2}\binom{n-1}{j+1} a^{j+1}(1-a)^{n-2-j} + 1\right\}\\
&= \underline{p}\left\{\frac{1-F(\underline{p}+c)}{a}\sum_{k=1}^{n-1}\binom{n-1}{k} a^{k}(1-a)^{n-1-k} + 1\right\}\\
&= \underline{p}\left\{\frac{1-F(\underline{p}+c)}{a}[1-(1-a)^{n-1}] + 1\right\},
\end{aligned}
$$

which has to be equal to V. However, when we approach $\underline{p}$ from the right, we have, for every p in its vicinity, using (1):

$$
\begin{aligned}
V &= E\pi_1(p, p_2, \ldots, p_n)\\
&= p\{1 + (n-1)[1 - F(p+c)][1 - F(p)]^{n-2}\}\\
&\xrightarrow[p > \underline{p}]{p \to \underline{p}} \underline{p}\{1 + (n-1)[1 - F(\underline{p}+c)](1-a)^{n-2}\}.
\end{aligned}
$$

The limit also equals V because F is continuous from the right. That gives the equation

$$
\frac{1 - F(\underline{p}+c)}{a}[1-(1-a)^{n-1}] + 1 = 1 + (n-1)[1 - F(\underline{p}+c)](1-a)^{n-2}
$$

or

$$
(n-1)\,a(1-a)^{n-2} + (1-a)^{n-1} = 1,
$$

which has the unique solution $a = 0$. So a jump at $\underline{p}$ is ruled out. From $F(\underline{p}) = 0$ we get through (3),

$$1 - [V/(\underline{p} + c)]^{1/(n-1)} = 0,$$

which gives

$$V = \underline{p} + c.$$

To be a solution, F has to satisfy the condition that it does not pay to deviate to $p > \bar{p}$ or $p < \underline{p}$.

For $p > \bar{p}$, that requires, using (1),

$$[1 - F(p - c)]^{n-1} \leq V/p,$$

and if $\bar{p} - \underline{p} < 2c$ then $F(p - c)$ is in the flat portion; so we want $(V/\bar{p})^{1/(n-1)} \leq V/p$, which is impossible since $V/p < V/\bar{p} < (V/\bar{p})^{1/(n-1)}$; hence when $\bar{p} - \underline{p} < 2c$, it must be that $\bar{p} = 1$. So, two cases have to be distinguished, the incidence of which depends on the size of c:

$$\text{(a)} \quad \bar{p} - \underline{p} \geq 2c.$$

Then we require

$$p\left(\frac{[V/(p - 2c)] - 1}{(n-1)[V/(p - c)]^{(n-2)/(n-1)}}\right)^{n-1} \leq V.$$

The left-hand side has a negative derivative, so it is enough to check for $p = \bar{p}$. Note that $\bar{p} - \underline{p} = 2c$ and $V = \underline{p} + c$ imply $\bar{p} = V + c$:

$$\begin{gathered} \bar{p}\left(\frac{[V/(\bar{p} - 2c)] - 1}{(n-1)[V/(\bar{p} - c)]^{(n-1)/(n-2)}}\right) = (V + c)\left(\frac{c/(V - c)}{n - 1}\right)^{n-1} \leq V, \\ \left(\frac{c}{(n-1)(V - c)}\right)^{n-1} \leq \frac{V}{V + c}. \end{gathered} \tag{5}$$

For $p < \underline{p}$ we want

$$p\{1 + (n - 1)[1 - F(p + c)]\} \leq V,$$

which, for $\bar{p} - \underline{p} \geq 2c$, is

$$p\{1 + (n - 1)[V/(p + 2c)]^{1/n-1}\} \leq V.$$

The left-hand side is increasing in p, so it is enough to require that for $p = \underline{p} = V - c$:

$$(V-c)\left\{1+(n-1)\left(\frac{V}{V+c}\right)^{1/n-1}\right\} \leq V,$$
$$\frac{V}{V+c} \leq \left[\frac{c}{(n-1)(V-c)}\right]^{n-1}. \tag{6}$$

Equations (5) and (6) together give an equation, writing subscripts n to make explicit the dependence of results on n:

$$\frac{V_n}{V_n+c} = \left[\frac{c}{(n-1)(V_n-c)}\right]^{n-1}. \tag{7}$$

The solution of (7) is a linear function of c, $V_n = a_n c$, where a_n is the solution of

$$\frac{a_n}{a_n+1} = \left[\frac{1}{(n-1)(a_n-1)}\right]^{n-1}.$$

which satisfies $a_n \rightarrow_{n\to\infty} 1$.

Selected values of a_n are

n	2	3	4	5	10	100	1000
a_n	2.414	1.635	1.398	1.289	1.119	1.010	1.001

So, in the case $\bar{p} - \underline{p} = 2c$, we have a solution with $V_n = a_n c$, $\underline{p}_n = a_n c - c$, $\bar{p}_n = a_n c + c$. F is continuous everywhere and has no flat segment.

Because prices are bounded, $\bar{p} \leqslant 1$, this case can apply only if $\bar{p} = a_n c + c \leqslant 1$ or $c \leqslant 1/(a_n + 1)$. When $1/(a_n+1) < c < (n-1)/n$, we have

(b) $\bar{p} - \underline{p} < 2c$.

In this case, $\bar{p} = 1$; so we need not worry about deviation to prices $p > \bar{p}$. Against the desirability of charging lower prices, $p < \underline{p}$, we require, as in case (a),

$$p\{1+(n-1)[1-F(p+c)]\} \leqslant V.$$

Now, however, since for small deviations $F(p+c)$ is in the flat segment, $F(p+c) = 1 - (V/\bar{p})^{1/n-1} = 1 - V^{1/n-1}$; so we want

$$p\{1+(n-1)\,V^{1/n-1}\} \leqslant V, \qquad \text{for} \quad \underline{p} - c \leqslant p < \underline{p};$$

it is enough to require it for $p = \underline{p} = V - c$,

$$(V - c)\{1 + (n - 1)\, V^{1/n-1}\} \leq V,$$

$$V \leq \left[\frac{c}{(n - 1)(V - c)}\right]^{n-1}. \tag{8}$$

The condition for a nonnegative jump of F at $p = \underline{p} + c = V$ is

$$1 - V^{1/n-1} \leq 1 - \frac{V/(V - c) - 1}{n - 1}$$

or

$$\left[\frac{c}{(n - 1)(V - c)}\right]^{n-1} \leq V. \tag{9}$$

Equations (8) and (9) together give

$$\left[\frac{c}{(n - 1)(V - c)}\right]^{n-1} = V. \tag{10}$$

Let us define $g_n(c)$ as the solution of (10), i.e.,

$$\left[\frac{c}{(n - 1)(g_n(c) - c)}\right]^{n-1} = g_n(c). \tag{10'}$$

$g_n(c)$ satisfies

$$g_n\left(\frac{1}{a_n + 1}\right) = \frac{a_n}{a_n + 1}, \qquad g_n\left(\frac{n - 1}{n}\right) = 1, \qquad \text{and} \qquad g'(c) > 0.$$

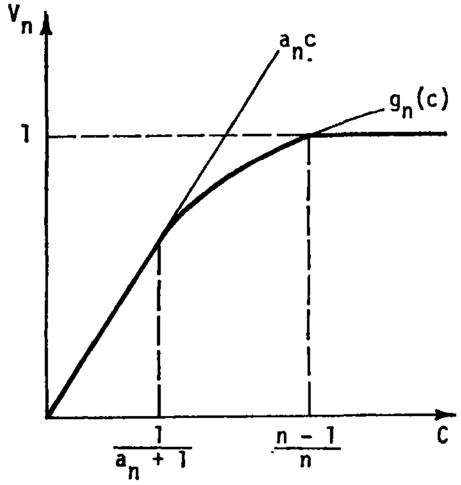

FIGURE 2

So, in case (b), when $1/(a_n + 1) < c < (n - 1)/n$, we have a solution with $V_n = g_n(c)$, $\underline{p}_n = g_n(c) - c$, $\bar{p}_n = 1$. F is continuous everywhere except at 1, where it manifests a jump of size

$$\frac{[g_n(c)/(1 - c)] - 1}{(n - 1)[g_n(c)]^{(n-2)/(n-1)}} > 0.$$

For $c > (n - 1)/n$, the markets are completely isolated and each seller can safely charge the monopolist price $p = 1$, because for a deviation to $p = 1 - c - \epsilon$ for some small $\epsilon > 0$ to be profitable, it is necessary that $n(1 - c - \epsilon) > 1$ or $[(n - 1)/n] - \epsilon > c$, contrary to $c > (n - 1)/n$. Figure 2 depicts V_n as a function of c. For every c there is a unique symmetrical solution.

4

It was assumed previously that $\bar{p} - \underline{p} < 2c$. Here we justify that assumption and prove that there are no solutions with $\bar{p} - \underline{p} > 2c$. The faithful reader may skip this section without loss of continuity.

Suppose $\bar{p} - \underline{p} > 3c$. Then from the difference equation (Eq. (1)),

$$[1 - F(p - c)]^{n-1} + (n - 1)[1 - F(p + c)][1 - F(p)]^{n-2} = V/p,$$

we get, by substituting $F(p + c)$ for $F(p)$ and $F(p - c)$,

$$n[1 - F(p + c)]^{n-1} \leq V/p. \tag{11}$$

Noting that, always, no matter what is the range $\bar{p} - \underline{p}$,

$$F(p) = 1 - [V/(p + c)]^{1/n-1}, \qquad \text{for} \quad \bar{p} - 2c < p < \bar{p} - c, \tag{3}$$

we get, from (11), by taking $p = \bar{p} - 2c - \epsilon$ for small $\epsilon > 0$,

$$nV/(\bar{p} - \epsilon) \leq V/(\bar{p} - 2c - \epsilon),$$
$$\bar{p} \leq [2n/(n - 1)]\, c + \epsilon;$$

hence

$$\bar{p} \leq [2n/(n - 1)]c, \tag{12}$$

which handily limits our inquiry since, for $n = 2$, $\bar{p} \leq 4c$; so $\bar{p} - \underline{p} < 4c$. $\bar{p} - \underline{p} = 4c$ implies $\underline{p} = 0$, which is impossible since the payoff is a multiplication of p, as can be seen in (1). For $n \geq 3$, $\bar{p} \leq 3c$ and $\bar{p} - \underline{p} < 3c$, contrary to the assumption $\bar{p} - \underline{p} > 3c$.

First, the case $n = 2$. Equation (1) has the form

$$2 - F(p - c) - F(p + c) = V/p. \tag{1'}$$

For the case $3c < \bar{p} - \underline{p} < 4c$, we get from (1′) that if a solution F exists, it has to satisfy

$$\begin{aligned}
F(p) &= 1 - V\{[1/(p + c)] - [1/(p + 3c)]\}, && \underline{p} \le p < \bar{p} - 3c,\\
&= 2 - [V/(p - c)], && \underline{p} + c \le p < \bar{p} - 2c,\\
&= 1 - [V/(p + c)], && \bar{p} - 2c \le p < \bar{p} - c,\\
&= V\{[1/(p - 3c)] - [1/(p - c)]\}, && \underline{p} + 3c \le p < \bar{p},\\
&= 1, && \bar{p} \le p.
\end{aligned}$$

We do not give F fully because it is not necessary: it does not exist anyway.

The proof given in Section 3 that $F(\underline{p}) = 0$ still holds for this case and implies now $V = (\underline{p} + c)(\underline{p} + 3c)/2c$.

The inequality $F(\bar{p} - 2c -) \le F(\bar{p} - 2c)$ gives

$$2 - \frac{V}{\bar{p} - 3c} \le 1 - \frac{V}{\bar{p} - c},$$

$$\frac{(\bar{p} - c)(\bar{p} - 3c)}{2c} \le V.$$

The inequality $F(\bar{p} -) \le F(\bar{p}) = 1$ gives

$$V\left[\frac{1}{\bar{p} - 3c} - \frac{1}{\bar{p} - c}\right] \le 1,$$

$$V \le \frac{(\bar{p} - c)(\bar{p} - 3c)}{2c};$$

so, together, we have

$$V = \frac{(\bar{p} - c)(\bar{p} - 3c)}{2c} = \frac{(\underline{p} + c)(\underline{p} + 3c)}{2c}.$$

This equality, however, holds if and only if $\bar{p} - \underline{p} = 4c$ or $\underline{p} = 0$, which, as noted, is impossible. For the case $2c < \bar{p} - \underline{p} < 3c$, (1) gives two different definitions of F in the range $\underline{p} + c < p < \bar{p} - c$.

Now take $n > 2$. The case $\bar{p} - \underline{p} = 3c$ can be dismissed since it is possible only for $n = 3$ and implies then $p = 0$. Assume now $2c < \bar{p} - p < 3c$. Using the basic difference equation (1), we get a candidate for a solution:

$$
F(p) = \begin{cases}
1 - \left[\dfrac{(V/p) - 1}{(n-1)[V/(p+2c)]^{1/n-1}}\right]^{1/n-2}, & p \le p \le \bar{p} - 2c, \\[2ex]
1 - [V/(p+c)]^{1/n-1}, & \bar{p} - 2c < p \le \bar{p} - c, \\[2ex]
1 - \dfrac{[V/(p-c)] - 1}{(n-1)(V/\bar{p})^{n-2/n-1}}, & \bar{p} - c < p \le p + 2c, \\[2ex]
1 - \dfrac{\dfrac{V}{p-c} - \left[\dfrac{[V/(p-2c)] - 1}{(n-1)(V/\bar{p})^{1/n-1}}\right]^{n-1/n-2}}{(n-1)(V/\bar{p})^{n-2/n-1}}, & p + 2c < p \le \bar{p}, \\[2ex]
1, & \bar{p} \le p.
\end{cases}
$$

F must be nondecreasing and that imposes, by a comparison of the two values at each of the switching points, two equations:

$$\frac{V}{p} = 1 + (n-1)\left(\frac{V}{p+2c}\right)^{1/n-1}, \tag{13}$$

$$\frac{V}{\bar{p} - c} = \left[\frac{[V/(\bar{p} - 2c)] - 1}{(n-1)(V/\bar{p})^{1/n-1}}\right]^{n-1/n-2}. \tag{14}$$

These two equations have no solution that satisfies

$$p + c < V < \bar{p} - c, \tag{15}$$

which was assumed in their construction, because if $V = p + c$, $F(p + 2c) = 1$ and if $V = \bar{p} - c$, $F(\bar{p} - 2c) = 0$. To show it, we rewrite the equations

$$V = p + (n-1)\,p[V/(p+2c)]^{1/n-1}, \tag{13'}$$

$$\bar{p} = c + V\left[\frac{(n-1)(V/\bar{p})^{1/n-1}}{[V/(\bar{p} - 2c)] - 1}\right]^{n-1/n-2}. \tag{14'}$$

Let us take $V = p + c$. Solving (13′) for it, we get

$$c = (n-1)\,p^*[(p^* + c)/(p^* + 2c)]^{1/n-1}.$$

Then we note that (14′) is solved by $\bar{p} = p^* + 2c$. So we have a solution $(p, V, \bar{p}) = (p^*, p^* + c, p^* + 2c)$ for the two equations, one that violates (15). From (13′) we have

$$\frac{dV}{dp} = \frac{1 + (n-1)[V/(p+2c)]^{1/n-1} - [p/(p+2c)][V/(p+2c)]^{1/n-1}}{1 - (p/V)[V/(p+2c)]^{1/n-1}} > 1,$$

When searching for another p, we certainly cannot decrease p below p^* because that will decrease $V - p$ below c. Can we then increase p, and V with it? By differentiating (14′) we have

$$\frac{d\bar{p}}{dV} = \frac{[V/(V + 2c - \bar{p})] - 1}{\left(\frac{V}{\bar{p} - 2c} - \frac{n-2}{n-1}\,\frac{V}{\bar{p} - c}\right) + \left(\frac{V}{V + 2c - \bar{p}} - \frac{V}{(n-1)\,\bar{p}}\right)} < 1,$$

which implies that by increasing p beyond p^*, V will increase and so will $\bar{p}$. But $\bar{p} - V$ will decrease, again a violation of (15). That completes the proof.

5

The Nash concept of solution does not guarantee, in general, stability in the following sense.[1] Examples can be concocted in which one player deviates from the Nash solution, and the other players react to that by treating him no longer as a player but as part of the environment of a new game and by doing so afford him a higher payoff than in the original game. That makes the incentive to deviate strong despite the immediate loss associated with it.

In our game, if a player deviates to a pure strategy, the new game has no solution at all because of the discontinuities involved. So deviation must be to a mixed strategy that is not too steep, if it exists. It also has to be a movement toward higher prices because players will react to being cut by lowering their own prices, thus decreasing the total pie.

However, even if the new game has a solution and deviation is profitable, this challenge to the Nash solution in our game is not without difficulties. If it is good to deviate, then why wouldn't every player try it? If many do, it becomes attractive to the remaining ones to cut them, eliminating the possibility of a solution with higher payoffs to the deviators. Also, when a player raises prices, the total pie increases; the other players then gain the loss of the deviator plus the addition to the pie and do not have immediate pressure to change their behavior.

So, although we have no answer to the question whether deviation may be profitable, it seems that a deliberate decision to leave the solution is not a viable threat to the solution in this game, while a casual or mistaken deviation is effectively policed by the Nash solution.

6. Concluding Remarks and Extensions

1. The source of the monopolistic power in this model is the consumers' being "stuck" in their segment of the market and incurring a cost in moving

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[1] L. S. Shapeley suggested this question to me.

out. As seen in Fig. 2, as c goes down, this monopolistic power evaporates and prices and payoffs go to zero. For $c \geq (n-1)/n$, the markets are completely separated, affording every seller a position of monopoly in his own segment.

2. As the number of players increases, the competition gets fiercer. The payoff for lower prices is now higher since consumers from more markets can be attracted and hence the whole range of charged prices has to go down. Indeed, from (7′) and (10′), we see that

$$V_n = \begin{cases} a_n c, & c \leq 1/(a_n+1), \\ g_n(c), & c \geq 1/(a_n+1), \end{cases} \xrightarrow[n\to\infty]{} c.$$

c is the lowest payoff in this game because a player can guarantee himself c by charging $p = c$. For prices up to c, every market is protected by the cost c, which acts as a tariff. When competition increases, payoffs approach this minimal level.

The increase of n in this game does not correspond to a simple entry of new competitors because, since the number of market segments increases as well, the total demand is not constant. Nevertheless, this result would suggest that in a large metropolitan area with many market segments and sellers, prices and payoffs would tend, ceteris paribus, to be lower.

It should be noted, however, that as n increases, F approaches a step function with a unit step at $p = c$. That is not a solution for the game with $\aleph_0$ players, which has no solution.

3. We shall now slightly modify the game to limit the total quantity to D. Also, fixed cost S is introduced and a zero profit condition is imposed. This way we shall get the number of sellers as an endogenous variable. It is assumed that the market is divided into n equal segments of D/n each; so when n increases, a new equal division takes place.

Denoting the value of the new game by u_n, we write the new basic difference equation:

$$p(D/n)\{[1-F(p-c)]^{n-1} + (n-1)[1-F(p)]^{n-2}[1-F(p+c)]\} = u_n. \quad (15)$$

We note that the solution F of the old game will also solve this one if we put $u_n = (D/n)\, V_n$. In this new game, new entrants drive the payoff u_n down to zero, but prices behave as before. Imposing $u_n - S = 0$ or $V_n = (nS/D)$ and using (7) and (10′), we can get n as a function of the parameters S, D, c:

$$n(S, D, C) = [X^*],$$

where X^* is the solution of

$$\frac{XS/D}{(XS/D)+c} = \left[\frac{c}{(X-1)[(XS/D)-c]}\right]^{X-1}, \qquad c \leq c^*(S, D),$$

$$XS/D = \left[\frac{c}{(X-1)[(XS/D)-c]}\right]^{X-1}, \qquad c \geq c^*(S, D),$$

and $c^*(S, D)$ is the solution of

$$1-c = \left[\frac{c}{[[(1-c)D/S]-1](1-2c)}\right]^{((1-c)D/S)-1}.$$

If X^* is not an integer, $[X^*]+1$ competing firms incur net loss, while $[X^*]$ firms have positive profit. So, instability is conceivable if these above-normal profits attract new entrants who do not assess the situation correctly.

As can be expected, we have $\Delta n/\Delta S \lesssim 0$, $\Delta n/\Delta D \gtrsim 0$, $\Delta n/\Delta c \lesssim 0$.

4. The segmentation of the market can take place for different reasons. The locational distribution of consumers and the travel cost between locations is one of them. Another one is the loyalty distribution of consumers among brands and the psychic cost of switching from one to another. An informational interpretation can also be given according to which consumers are segmented in regard to their free access to information. Suppose every consumer has free information about the price in one store, the one he sees from his window or the one he happened to visit in the course of buying another commodity. He also knows from the newspaper or other published material the range of prices charged and, in particular, the lowest price; but he does not know which store has the lowest price and to find out costs c.

5. The game analyzed in this paper is highly discontinuous; however, it seems from partial results not reported here that the inexistence of pure-strategy equilibrium and hence the necessary resort to randomization are more general and appear in continuous versions as well.

6. It is hard to believe that firms go through all the calculations and compute the solving price-distribution to be followed. The solution become impossibly complicated when players are not identical, as is usually the case. However, a solution may be attempted by shrewd sellers who "feel" it in their guts with higher or lower degrees of accuracy, and those who do not, lose out.

No adjustment process can be expected to lead to the Nash solution outlined here or to any Nash solution for that matter. Indeed, the two represent entirely different philosophies. In an adjustment process, an actor responds to his environment in an optimizing fashion, while in a game no player can take the other players as part of the environment, since they are active and rational players just as himself.

7. For small c, when $\bar{p} - p \quad 2c$, we have $\bar{p} < 1$, so the constraint $p \cdot 1$ is not binding. That means that the solution we have is also the solution for the same game with a higher bound on prices.[2] Economically it means that even when consumers are ready to pay any price for the good, the competitive nature of the game effectively puts a check on the sellers' behavior and limits their prices.

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[2] D. Gale pointed this out to me.

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Characterizing Optimal Policies in a One-Sector Model of Economic Growth Under Uncertainty*

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INTRODUCTION

The steady-state behavior of growing economies under the influence of an optimal policy in an uncertain environment turns out to have a character which is significantly different in many ways than its counterpart in the deterministic theory. It was originally shown by Brock and Mirman [1] in a one-sector model that the optimal steady state of the capital stock consists of a stationary distribution for the Markov process of capital accumulation. Moreover, while assuming that for each positive input the range of possible outputs is a compact interval bounded away from zero (in particular the minimum possible output for each positive input is positive) and that the usual neoclassical assumptions of growth theory hold, it was shown, in the special case of a finite number of states of the world,[1] that the support of the stationary distribution is a compact interval, bounded away from zero. Hence the optimal steady-state consumption is constrained to a compact interval which is also bounded away from zero. If this result were generally valid it would imply that the price function associated with the optimal steady state must be bounded. Note that in the deterministic case, at the unique optimal steady state, the price, which is the marginal utility of optimal consumption,

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[1] This result of Brock and Mirman is also valid if the probability of the "smallest" output is always positive (see [1, 10]).

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is in fact a well-defined positive number. However, under uncertainty this need not be the case. In fact it was shown by Mirman and Zilcha [10] that there exists an example in which the "shadow" price function associated with the optimal steady state (which under uncertainty is a function of the exogeneous random variable) is not bounded. This is implied, as shown in Mirman and Zilcha, by the fact that in this example the steady-state distribution of the capital stock is supported on a compact interval containing the origin. The example shows that although zero capital stock carries zero probability, it is not enough to imply that the price function is essentially bounded. The boundedness of the price function was used in [2] to show that finite horizon competitive programs or optimal finite horizon programs converge, as the horizon tends to infinity, to an infinite horizon competitive program which is optimal. The proof of Brock and Mirman in [2] actually used the boundedness of prices. If the optimal steady state is not bounded away from zero a problem arises in the Brock–Mirman proofs in [2] since it is no longer obvious that the expected value of the capital stock is bounded.

In this paper a different approach will be taken. In this approach it is not necessary that the price function be bounded. The strategy is to define a valuation function and derive the functional equation rigorously. The results obtained by Radner in [13], i.e., the existence of an optimal stationary program and its characterization by an integrable price function, will be used extensively in this paper, since programs will be compared by their relationship to the utility of the optimal stationary consumption policy. It should be noted that in place of a finite price function the approach of this paper uses only the finiteness of the *expected* prices.

We define a valuation function in the nondiscounting case, in a similar way to that in [14]. Using a dynamic programming argument a rigorous foundation for the functional equation approach used by Brock and Mirman [2] is presented. This leads naturally to two immediate results. First, the example of Mirman and Zilcha in [10] holds in the no-discounting model. Second, the important fact [2, Lemmas 1.1 and 1.2] that under an optimal policy, whenever the capital stock is positive, the consumption is positive follows simply without the additional assumption that the production functions are ordered in the random variable.

The main result of this paper is that the transversality condition, in its expected value version, is a necessary and sufficient condition for a competitive program to be optimal. We generalize the well-known condition under certainty (see Peleg [12]) which enables us to attain an optimal allocation through a system of decentralized decision-making by appropriately using the supporting prices (see [7]). The necessity of the transversality condition demonstrates that one may not be able to dismiss this basic economic issue by simply conjecturing that the introduction of uncertainty about future technology and risk aversion on the part of the producers would promptly close off the loophole (see Koopmans [7, p. 112]). We prove that, although the competitive price system of the optimal program need not be bounded, it is bounded in the mean. This fact is true for any competitive program whenever the transversality condition holds.

THE MODEL

We use the model of Brock and Mirman [2] without making the same restrictive assumptions as in [2]. In particular, we do not assume that $f(x, r)$ is monotone in r for each $x > 0$. This assumption, that the production functions are ordered, is very restrictive from the economic point of view (see [9]). We shall briefly present the model. Let y_0 be the initial capital stock and

$$\begin{aligned} x_0 + c_0 &= y_0, \qquad c_t \geq 0, \qquad x_t \geq 0, \\ x_{t+1} + c_{t+1} &= f(x_t, r_{t+1}), \qquad t = 0, 1, \ldots, \end{aligned} \tag{1}$$

where x_t is capital stock per capita. The production function is given in the per capita form $f(x, r)$, where r is a random variable. For each value of r, $f(\cdot, r)$ is increasing, strictly concave, and continuously differentiable with $f'(0, r) > 1$ and $f'(\infty, r) < 1$, for all r. The random variables r_t, $t = 0, 1, \ldots$ are independent, identically distributed, and take values in $[\alpha, \beta]$, where $0 < \alpha < \beta < \infty$.[2] Let ν be the probability measure on the Borel sets of $[\alpha, \beta]$ which is determined by the common distribution function of the r_t's.

Let $\mathbf{x} = (x_0, x_1, \ldots)$, $\mathbf{c} = (c_0, c_1, \ldots)$ satisfy (1) for $t = 0, 1, \ldots$. Let $y_t = f(x_{t-1}, r_t)$, $t = 1, 2, \ldots$ and $\mathbf{y} = (y_0, y_1, \ldots)$. The triplet $(\mathbf{x}, \mathbf{y}, \mathbf{c})$ is called a *feasible program* from initial stock y_0. Denote by $P(y_0)$ the set of all feasible programs from y_0. Let u be the utility function defined on the consumption space, $u: E_+^1 \to E^1$. We assume that u is increasing strictly concave, continuously differentiable and that $u'(0) = \infty$ (we allow $u(0) = -\infty$). Since future utilities are not discounted the comparison of feasible programs is done by maximizing the expected "sum" of nondiscounted utilities in the sense of the "catching-up" criterion used by Gale in [5]. The program $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{c}^*)$ is said to be *optimal* in $P(y_0)$ if for any $(\mathbf{x}, \mathbf{y}, \mathbf{c})$ in $P(y_0)$, $\lim_{N \to \infty} \sup \sum_{t=0}^{N} [Eu(c_t) - Eu(c_t^*)] \leq 0$.

Let $\mathscr{B}$ be the Borel sigma field in $[\alpha, \beta]$. Let $R = ([\alpha, \beta], \mathscr{B}, \nu)$. Define the product probability space $(S, \mathscr{S}, \sigma) = \mathsf{X}_{t=-\infty}^{\infty} R_t$, where $R_t = R$ for $-\infty < t < \infty$. The sigma field $\mathscr{S}_t$ is generated by cylinder sets $\mathsf{X}_{-\infty}^{\infty} B_k$, where $B_k = [\alpha, \beta]$, for all $k > t$. The set of all real-valued $\mathscr{S}_t$-measurable essentially bounded functions on S is denoted by $L_\infty(S, \mathscr{S}_t, \sigma)$ (in the sequel the notation $L_\infty(\mathscr{S}_t)$ will be used). Let $L_\infty^+(\mathscr{S}_t)$ be the nonnegative cone of $L_\infty(\mathscr{S}_t)$. $T: S \to S$

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[2] Here it is assumed that $0 < f(x, r) < \infty$ for each $x > 0$ and $r \in [\alpha, \beta]$.

is the shift operator defined by $(Ts)_t = s_{t+1}$ for all $-\infty < t < \infty$ and s in S. It is clear that T and T^{-1} are measure preserving. For f in $L_\infty(\mathcal{S}_t)$ define Tf by $Tf(s) = f(Ts)$ for all s in S.

A *technologically feasible stationary program* is defined in the same way as was defined by Radner [13]. The pair (x, y) is *technologically feasible* if x is in $L_\infty(\mathcal{S}_0)$, y is in $L_\infty(\mathcal{S}_1)$, and $y(s) = f(x(s), s_1)$ a.s. Let the set of all the technologically feasible stationary programs be denoted by G.

A triplet (x, y, c) is a *feasible* stationary program if $(x, y) \in G$ and $c(s) = y(T^{-1}s) - x(s) \geq 0$ a.s.; namely, feasible consumption is almost surely nonnegative.

Remark. It is clear that if $(\mathbf{x}, \mathbf{y}, \mathbf{c})$ is in $P(y_0)$ then for all t, $t = 0, 1, 2, \ldots$, (x_t, y_{t+1}) is in T^tG, and $c_t(s) = y_t(s) - x_t(s) \geq 0$ a.s.

The important results derived by Radner [13] in the stationary case will be used throughout this paper. Radner considers a model containing both produced and primary resources. In the one-sector model considered in this paper labor is the only primary good. Moreover, with the per capita production function $f(x, r)$, the primary resource supplied is $w(s) = 1$ a.s. The input–output possibility sets in the one-sector model are given by

$$G(s) = \{(x, y) \mid 0 \leq y \leq f(x, s_1)\}.$$

Since f is the usual neoclassical production function, Radner's assumptions in [13] about $G(s)$ are satisfied, i.e., $G(s)$ is closed convex with vertex 0. Also, free disposal is implicit. Finally, primary inputs are necessary and are not producible.

Radner proved in [13] that there exists an *optimal stationary program* (x^*, y^*, c^*), i.e.,

$$\int u(c^*)\, d\sigma = \sup\left\{\int u(c)\, d\sigma \mid (x, y) \text{ in } G \text{ and } c = T^{-1}y - x \geq 0\right\},$$

u is strictly concave, hence it is unique. Let the optimal stationary expected utility be denoted by $v^* = Eu(c^*)$.

THEOREM (Radner). *If u is increasing, there exists a price function ψ in $L_1(\mathcal{S}_0)$, $\psi(s) > 0$ a.s. such that*

$$\int u(c^*)\, d\sigma - \int \psi c^*\, d\sigma \geq \int u(c)\, d\sigma - \int \psi c\, d\sigma \tag{2}$$

for all c in $L_\infty^+(\mathcal{S}_0)$, and

$$\int T\psi y^*\, d\sigma - \int \psi x^*\, d\sigma \geq \int T\psi \cdot f(x, s_1)\, d\sigma - \int \psi x\, d\sigma \tag{3}$$

for any x in $L_\infty^+(\mathcal{S}_0)$.

(See [13, Theorem 5.1] and [3, Theorem 8.1].) Inequalities (2) and (3) will be used below to derive a valuation function. The program (**x**, **y**, **c**) in $P(y_0)$ is a *good* program if there is a constant M such that $\sum_{t=0}^{N} [Eu(c_t) - v^*] \geq M$, $N = 1, 2, \ldots$. If (**x**, **y**, **c**) is not a good program then $\sum_{t=0}^{\infty} [Eu(c_t) - v^*] = -\infty$ (see [6, Theorem 7]).

LEMMA 1. *If* (**x**, **y**, **c**) *is a good program in* $P(y_0)$ *then* $\sum_{t=0}^{\infty} [Eu(c_t) - v^*]$ *converges.*

Proof.

$$
\begin{aligned}
\sum_{t=0}^{N} [Eu(c_t) - Eu(c^*)] \\
&= \sum_{t=0}^{N} [Eu(c_t) - ET^t\psi c_t - (Eu(c^*) - E\psi c^*)] \\
&\quad + \sum_{t=0}^{N} [ET^t\psi(y_t - x_t) - E(T\psi y^* - \psi x^*)] \\
&= \sum_{t=0}^{N} [Eu(c_t) - ET^t\psi c_t - (Eu(c^*) - E\psi c^*)] \\
&\quad + \sum_{t=0}^{N} [ET^{t+1}\psi y_{t+1} - ET^t\psi x_t - (ET\psi y^* - E\psi x^*]) \\
&\quad + E\psi y_0 - ET^{N+1}\psi y_{N+1}.
\end{aligned}
\tag{4}
$$

Since (**x**, **y**, **c**) is a good program, $\sum_{t=0}^{N} [Eu(c_t) - Eu(c^*)]$ is bounded from below. It is bounded from above for all N by $E\psi y_0$. This can be verified since inequalities (2) and (3) imply that the two summations are nonpositive, therefore these partial sums converge (note that T and T^{-1} are measure preserving). In particular, from (4) it is easy to see that for any good program (**x**, **y**, **c**),

$$\int T^{t+1}\psi y_{t+1} - \int T^t\psi x_t \to \int T\psi y^* - \int \psi x^*, \qquad \text{as} \quad t \to \infty. \tag{*}$$

By the assumptions on the production function it is clear that there exists a compact set K such that for any feasible program (**x**, **y**, **c**), $x_t \in K$ a.s., $y_{t+1} \in K$ a.s. and $c_t \in K$ a.s. for all $t = 0, 1, \ldots$.

Consider now the sequence $\{(T^{-t}x_t, T^{-t}y_{t+1})\}$ which is derived from the good program (**x**, **y**, **c**). The compactness argument implies that the sequence $\{(T^{-t}x_t, T^{-t}y_{t+1})\}$ is relatively compact as a subset of $L_1(\mathscr{S}_0) \times L_1(\mathscr{S}_1)$ (see [11, Neveu Theorem IV.2.3]). Therefore, it is weakly sequentially compact

([4, Theorem V.6.1]). Hence to the limit $(\hat{x}, \hat{y})$ of any subsequence $\{T^{-t_k} \cdot x_{t_k}, T^{-t_k} \cdot y_{t_k+1}\}$ which converges in the weak topology of $L_1(\mathscr{S}_0) \times L_1(\mathscr{S}_1)$ corresponds the program $(\hat{x}, \hat{y}, \hat{c})$ which is a feasible stationary program. It is easy to see by the compactness that the above convergence is also in the weak* topology of $L_\infty(\mathscr{S}_0) \times L_\infty(\mathscr{S}_1)$. Since $(\hat{x}, \hat{y})$ is the limit of a subsequence of a good program it follows from (*) that

$$\int T\psi\hat{y} - \int \psi\hat{x} = \int T\psi y^* - \int \psi x^*. \tag{5}$$

Since $f(x, \eta)$ is almost surely strictly concave it follows from (5) that $\hat{x} = x^*$, otherwise define $\bar{x} = \frac{1}{2}(\hat{x} + x^*)$ then if $\hat{x} \neq x^*$ on a set of positive measure we see that

$$\int T\psi f(\bar{x}, s) - \int \psi\bar{x} > \int T\psi[\tfrac{1}{2}f(\hat{x}, s) + \tfrac{1}{2}f(x^*, s)] - \int \psi(\tfrac{1}{2}\hat{x} + \tfrac{1}{2}x^*)$$
$$= \int T\psi \cdot y^* - \int \psi x^*,$$

which is a contradiction to (3). This is true for any subsequence $T^{-t_k} \cdot x_{t_k}$ derived from the good program (**x**, **y**, **c**) converging in the weak* topology of $L_\infty(\mathscr{S}_0)$. Therefore $T^{-t}x_t$ converges to x^* in the weak* topology (which is a *metric* topology on bounded subsets of L_∞; see [4, Theorem V. 4.2]). Also, since u is strictly concave if $\hat{y} \neq y^*$, a contradiction to the maximality of $\int u(c^*)$ can be derived. Therefore $T^{-t}y_t$ converges weak* to y^*. Hence, $ET^{N+1}\psi y_{N+1} \to ET\psi y^*$ as $N \to \infty$. This completes the proof of the Lemma.

Define a valuation function ϕ on $L_\infty^+(\mathscr{S})$ in the following way:

$$\phi(y_0) = \sup\left\{\sum_{t=0}^{\infty} [Eu(c_t) - v^*] \mid (\mathbf{x}, \mathbf{y}, \mathbf{c}) \text{ in } P(y_0)\right\}.$$

It is clear that $\phi(y_0) > -\infty$ if and only if there is a good program in $P(y_0)$. Moreover it is easy to see that if $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{c}^*)$ is optimal in $P(y_0)$ then

$$\phi(y_0) = \sum_{t=0}^{\infty} [Eu(c_t^*) - v^*]. \tag{6}$$

For each y in $L_\infty^+(\mathscr{S}_0)$ if $y(s) > 0$ a.s. then $\phi(y) > -\infty$, i.e., there exists a good program in $P(y)$ (see [3, Theorem 9.2]). Moreover, ϕ is an increasing and strictly concave function (and hence continuous on $(0, \infty)$). These properties of the valuation function ϕ can be proved directly from those of the utility function u.

Using a dynamic programming argument it is clear that for any $x > 0$

$$\phi(x) = \max_{0 \leq c \leq x} \{u(c) - v^* + E\phi[f(x - c, \eta)]\}. \tag{7}$$

Since the maximand is continuous this maximum is obtained at a point $c = g(x)$ and $g(x) > 0$ since $u'(0) = \infty$. The maximand is strictly concave, hence $g(x)$ is uniquely defined by (7). The function g also has the important properties, $g(0) = 0$ and $g(x) > 0$, for all $x > 0$, which are not as easily derived using other methods (compare this with [2, Lemma 1.1]). The function $g(x)$ is called the *optimal policy function* (OPF). For all $x > 0$, $g(x)$ satisfies the functional equation,

$$\phi(x) = u(g(x)) - v^* + E\phi[f(x - g(x), \eta)]. \tag{8}$$

From (8) it can be verified that $g(x)$ is an increasing and continuous function. The triplet $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{c}^*)$ is optimal in $P(y)$ if and only if $c_0^* = g(y)$, $x_0^* = y - c_0^*$, and $c_t^* = g(f(x_{t-1}^*, s_t))$ for all $t \geq 1$.

LEMMA 2. *For all* $x > 0$, $\phi'(x)$ *exists and* $\phi'(x) = u'(g(x))$

Proof. Let $\bar{x}$ be any positive real number, and let $(\bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{\mathbf{c}})$ be the optimal program in $P(\bar{x})$. Since $g(x)$ is the OPF we have $\bar{c}_0 = g(\bar{x})$. Let $\Delta x > 0$. Define a feasible program from $\bar{x} + \Delta x$ as follows] $c_0 = \bar{c}_0 + \Delta x$, $c_t = \bar{c}_t$ for $t = 1, 2, \ldots$. It is clear that

$$\phi(\bar{x} + \Delta x) - \phi(\bar{x}) \geq \sum_{t=0}^{T} [Eu(c_t) - v^*] - \sum_{t=0}^{T} [Eu(\bar{c}_t) - v^*]$$

$$= u(\bar{c}_0 + \Delta x) - u(\bar{c}_0) = u'(\bar{c}_0)\,\Delta x + o(\Delta x).$$

Therefore $\phi_+'(x) \geq u'(\bar{c}_0) = u'(g(\bar{x}))$.

Now take Δx sufficiently small such that $0 < \Delta x < \bar{c}_0$. Define a feasible consumption program from $\bar{x} - \Delta x$ as follows: $c_0 = \bar{c}_0 - \Delta x$, $c_t = \bar{c}_t$ for $t \geq 1$. Using the same arguments as above, we find that

$$\phi(\bar{x} - \Delta x) - \phi(\bar{x}) \geq u(\bar{c}_0 - \Delta x) - u(\bar{c}_0) \geq -u'(\bar{c}_0)\,\Delta x + o(\Delta x).$$

Hence $\phi_-'(\bar{x}) \leq u'(\bar{c}_0) = u'(g(\bar{x}))$. Since ϕ is concave we have $\phi'(\bar{x}) = u'(g(\bar{x}))$ for any $\bar{x} > 0$.

Differentiating the maximand in (7) we have

$$0 = u'(g(x)) - E\{f'(x - g(x), \eta)\,\phi'[f(x - g(x), \eta)]\}.$$

The result of Lemma 2 yields

$$u'(g(x)) = \int f'(x - g(x), \eta)\, u'[g(f(x - g(x), \eta))]\,\nu(d\eta). \tag{9}$$

Hence we proved that this important functional equation holds in the nondiscounted models. In [2] Brock and Mirman used a completely different approach to achieve (9). Our approach proved to be much more useful, since the fact that $g(x) > 0$ for all $x > 0$ is a trivial consequence here, while in [2] Brock and Mirman had to work very hard to get this fact, using the additional assumption that $f(x, \eta)$ is monotone in η.

The functional equation (9) can be used to compute an OPF and therefore the steady-state probability distribution, just like the discounted case (see [9]). The analysis used to achieve (9) does not depend on the assumption that $f(x, \eta)$ is strictly concave in x, as is shown by the following examples.

EXAMPLE 1. Take $f(x, r) = rx$ and $u(c) = c^{1-\gamma}/(1-\gamma)$, where $\gamma > 0$ and suppose that the condition $\int_\alpha^\beta r^{1-\gamma}\,\nu(dr) < 1$ is satisfied. Putting $g(x) = \lambda x$ in (9) we find that (compare this with the example of Levhari and Srinivasan [8] in the discounted case) $(1-\lambda)^\gamma = Er^{1-\gamma}$. Hence, the OPF is $g(x) = [1-(Er^{1-\gamma})^{1/\gamma}]x$. Since $\phi'(x) = g(x)^{-\gamma}$ the valuation function is

$$\phi(x) = [1-(Er^{1-\gamma})^{1/\gamma}]^{-\gamma}[x^{-\gamma+1}/(-\gamma+1)].$$

Remark. In the linear case we see that $c^* = 0$. The above conditions guarantee that the optimal process $\{x_t^*\}$ converges to 0.

EXAMPLE 2. Assume that $f(x, r) = ax^r$, $a > 0$ and $0 < \alpha \leqslant r \leqslant \beta \leqslant 1$. Let $u(c) = \ln c$. Using (9) we find (just as in the discounted case in [9]) that the OPF is $g(x) = (1-\bar{\rho})x$, where $\bar{\rho} = \int_\alpha^\beta r\nu(dr)$. Hence the valuation function is

$$\phi(x) = [1/(1-\bar{\rho})] \ln x.$$

LEMMA 3. *Let* (**x**, **y**, **c**) *be any good program, and* (x^*, y^*, c^*) *be the optimal stationary program, then* $T^{-t}x_t \rightarrow_{t\to\infty} x^*$ *a.s. and* $T^{-t}y_t \rightarrow_{t\to\infty} y^*$ *a.s.*

Proof. From the proof of Lemma 1 we know that $T^{-t}c_t$ converges to c^* in the weak* topology of L_∞. From (2) it can be shown that (see [14]) $u(c^*(s)) - \psi(s)\,c^*(s) \geqslant u(c) - \psi(s)c$, a.s. for all $c \geqslant 0$. Since u is strictly concave we have strict inequality whenever $c \neq c^*(s)$. Now assume that c_t does not converge a.s. to c^*. There is $\epsilon > 0$ such that for some subsequence $\{c_{t_j}\}$ we have ($\psi(s) > 0$ a.s.)

$$Eu(c^*) - E\psi c^* \geqslant Eu(c_{t_j}) - ET^{t_j}\psi c_{t_j} + \epsilon.$$

But, observing (4), this implies that $\sum_{t=0}^\infty [Eu(c_t) - Eu(c^*)] = -\infty$, a contradiction.

For all η, $f(x, \eta)$ is monotone increasing and continuous. The OPF $g(x)$

is increasing and continuous, therefore $g(f(\cdot, s))$ is increasing and continuous for all s. Hence, since

$$g(f(T^{-t}x_t, s)) \xrightarrow[t\to\infty]{} g(f(x^*, s)) \qquad \text{a.s.}$$

it must be the case that $T^{-t}x_t \to x^*$, a.s., as $t \to \infty$. This implies that $T^{-t}y_t \to y^*$ a.s., which completes the proof.

Denote by $c_{t+1}(\eta) = g[f(\bar{x}_t, \eta)]$. A feasible program $(\mathbf{x}, \mathbf{y}, \mathbf{c})$ is called *competitive* if it satisfies

$$u'(\bar{c}_t) = \int_\alpha^\beta f'(\bar{x}_t, \eta)\, u'(\bar{c}_{t+1}(\eta))\, \nu(d\eta) \qquad \text{for} \quad t = 0, 1, 2,\dots . \tag{10}$$

In the deterministic one-sector model it is known (see [12]) that a competitive program is optimal if and only if it satisfies the transversality condition; i.e., the capital value is bounded. This is also proved in the stochastic discounted case when $u(0) \neq -\infty$ (see [9]). We shall generalize this important characterization to the nondiscounted stochastic case with no restriction on $u(0)$. This proof is also valid when future utilities are discounted.

THEOREM. *Let* $(\bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{\mathbf{c}})$ *be a competitive program from initial stock* $y_0 > 0$. *A necessary and sufficient condition that it is optimal is that there exists a constant* M *such that*

$$Eu'(\bar{c}_t)\, \bar{y}_t \leq M \qquad \textit{for all} \quad t. \tag{11}$$

Remark. In [2] Brock and Mirman proved that the good, competitive program which is the limit of finite horizon optimal programs is optimal. In proving this result they took it for granted that $\{\bar{c}_t\}_{t=0}^\infty$ is bounded away from zero. However, this is not necessarily true, as was shown in [10]. We shall prove that in the optimal case, although the prices $\{u'(\bar{c}_t)\}$ might not be bounded, the sequence of means of the prices $\{\int u'(\bar{c}_t)\}_{t=0}^\infty$ is bounded. The transversality condition (11) which holds for optimal policies also implies the Brock–Mirman results.

Proof of the Theorem. Suppose that $(\bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{\mathbf{c}})$ satisfies conditions (10) and (11). Denoting $\psi_t = u'(\bar{c}_t)$, condition (7) becomes

$$\psi_t = \int_\alpha^\beta f'(\bar{x}_t, \eta)\, \psi_{t+1}(\eta)\, \nu(d\eta) \qquad \text{for} \quad t = 0, 1,\dots .$$

Since f is concave

$$f'(\bar{x}_t, \eta) \leqslant [f(\bar{x}_t, \eta) - f(x, \eta)]/(\bar{x}_t - x) \qquad \text{for all} \quad x > 0.$$

Hence from (10) it follows that

$$\int \psi_{t-1} f(\bar{x}_t, s_{t-1}) - \int \psi_t \bar{x}_t \geq \int \psi_{t-1} f(x, s_{t-1}) - \int \psi_t x \tag{12}$$

for all $x \geq 0$, $t = 0, 1, 2, \ldots$.

From (11) it is clear that $\int \psi_t \bar{x}_t \leq M$ for all t, hence from (12) there exists a constant c such that

$$\int \psi_{t-1} f(x, s_{t-1}) - \int \psi_t x \leq c \quad \text{for all} \quad x \geq 0, \quad t = 0, 1, \ldots. \tag{13}$$

Let $\bar{x} > 0$ satisfy $f(\bar{x}, \eta) \geq 2\bar{x}$ for all η in $[\alpha, \beta]$. Define the sets A and B as

$$A = \left\{ t \,\Big|\, \int \psi_t \bar{x} \leq c \right\} \quad \text{and} \quad B = \left\{ t \,\Big|\, \int \psi_t \bar{x} > c \right\}.$$

Since $\bar{x}$ is a positive constant it is clear that for all t in A, $\int \psi_t \leq K$ for some constant K. Let $\bar{K} = \max \{K, \int \psi_0\}$ (if $A = \phi$, take $\bar{K} = \int \psi_0$). Let t_0 be any member of B; we shall prove that $\int \psi_{t_0} \leq \bar{K}$. From (13) it is clear that $\int \psi_{t_0} \cdot 2\bar{x} - \int \psi_{t_0-1} \bar{x} \leq c$, for $t_0 \in B$, therefore $\int \psi_{t_0} \bar{x} - \int \psi_{t_0-1} \bar{x} \leq 0$, which implies $\int \psi_{t_0} \leq \int \psi_{t_0-1}$. If $t_0 - 1 \in A$ we are finished, otherwise by the same argument $\int \psi_{t_0-1} \leq \int \psi_{t_0-2}$. This obviously leads to the conclusion that $\int \psi_{t_0} \leq \bar{K}$, in fact $\int \psi_t \leq \bar{K}$ for all t.

To show that $(\bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{\mathbf{c}})$ is optimal let $(\mathbf{x}, \mathbf{y}, \mathbf{c})$ be any good program in $P(y_0)$. From the concavity of $f(\cdot, \eta)$ and u we obtain the following inequalities using (10):

$$\begin{aligned}
&\sum_{t=0}^{N} [Eu(c_t) - Eu(\bar{c}_t)] \\
&\quad \leq \sum_{t=0}^{N} Eu'(\bar{c}_t)(c_t - \bar{c}_t) \\
&\quad \leq \sum_{t=0}^{N} Eu'(\bar{c}_t)\{f'(\bar{x}_{t-1}, s_t)[x_{t-1} - \bar{x}_{t-1}] - (x_t - \bar{x}_t)\} \\
&\quad = \sum_{t=0}^{N-1} E \left\{ \left[\int u'(\bar{c}_{t+1}(\eta)) f'(\bar{x}_t, \eta)\, \nu(d\eta) - u'(\bar{c}_t) \right] (x_t - \bar{x}_t) \right\} \\
&\qquad + Eu'(\bar{c}_N)(\bar{x}_N - x_N) \leq \int u'(\bar{c}_N) \cdot \operatorname{ess\,sup} | \bar{x}_N - x_N |.
\end{aligned}$$

It follows that any competitive program $(\bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{\mathbf{c}})$ which satisfies the transversality condition (11) is a good program, since for all N

$$\begin{aligned}
\sum_{t=0}^{N} [Eu(c_t) - Eu(\bar{c}_t)] &\leq Eu'(\bar{c}_N)(\bar{x}_N - x_N) \\
&\leq Eu'(\bar{c}_N)\, \bar{x}_N \leq K.
\end{aligned}$$

Since $(\mathbf{x}, \mathbf{y}, \mathbf{c})$ is good, $(\bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{\mathbf{c}})$ is good. Furthermore

$$\sum_{t=0}^{N} [Eu(c_t) - Eu(\bar{c}_t)] \leq Eu'(\bar{c}_N)(\bar{x}_N - x_N)$$
$$\leq \operatorname{ess\,sup} | \bar{x}_N - x_N | \cdot Eu'(\bar{c}_N) \to 0 \quad \text{as } N \to \infty,$$

since by Lemma 3 ess sup $| \bar{x}_N - x_N | \to 0$ as $N \to \infty$ and $Eu'(\bar{c}_N) < \bar{K}$ for all N. Hence

$$\limsup_{N\to\infty} \sum_{t=0}^{N} [Eu(c_t) - Eu(\bar{c}_t)] \leq 0,$$

and the optimality is established.

Necessity. Suppose $(\bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{\mathbf{c}})$ is the optimal program in $P(y_0)$. Let $h(x) = x - g(x)$. For any $x > 0$ we have $\min_{\alpha \leq \eta \leq \beta} h[f(x, \eta)] > 0$. This implies that for all t, $t = 0, 1, 2,\ldots$, $\bar{x}_t(s) > 0$ for all s. Hence $\bar{c}_t(s) > 0$ for all s, and all t. Put $x = f(\bar{x}_{t-1}, s_t)$ (which is positive for any finite t), in (9) we obtain (10). Therefore the optimal program is competitive.

For each t, $t = 1, 2,\ldots$ define $\phi_t(y) = \phi(T^{-t}y)$ for all y in $L_\infty^+(\mathscr{S}_t)$. Each ϕ_t has the same properties as ϕ. To prove that the transversality condition (11) is satisfied we shall use some results obtained in the proof of [14, Theorem 1]. There exists a sequence of continuous linear functionals π_t, $\pi_t \in L_\infty^*(\mathscr{S}_t)$, $\pi_t \geq 0$ for $t = 0, 1, 2,\ldots$ such that for all t,

$$\phi_t(\bar{y}_t) - \pi_t(\bar{y}_t) \geq \phi_t(y) - \pi_t(y) \qquad \text{for all } y \text{ in } L_\infty^+(\mathscr{S}_t). \tag{14}$$

There is a unique ψ_t in $L_1(\mathscr{S}_t)$, $\psi_t \geq 0$ such that $\pi_t(f) \geq \int \psi_t f\, d\sigma$, for all f in $L_\infty^+(\mathscr{S}_t)$, $t = 0, 1,\ldots$ while $\psi_t(s) = u'(\bar{c}_t(s))$ a.s. These $\{\pi_t\}$ were obtained by Zilcha [14] in a multisector model, and competitive prices for optimal programs (which are integrable functions) were derived from these continuous linear functionals. In our case $u'(\bar{c}_t(s))$ (which is in $L_1(\mathscr{S}_t)$) corresponds to the "absolutely continuous" component of π_t (see [4, p. 163]). We shall use these subgradients π_t as follows. Put $y = \frac{1}{2}\bar{y}_t$ in (14), then

$$\phi_t(\bar{y}_t) - \pi_t(\bar{y}_t) \geq \phi_t(\tfrac{1}{2}\bar{y}_t) - \pi_t(\tfrac{1}{2}\bar{y}_t).$$

Hence

$$\tfrac{1}{2}\pi_t(\bar{y}_t) \leq \phi_t(\bar{y}_t) - \phi_t(\tfrac{1}{2}\bar{y}_t) \qquad t = 0, 1, 2\ldots.$$

We shall prove that the right side of these inequalities is uniformly bounded. Since

$$\phi_0(y_0) = \sum_{t=0}^{N} [Eu(\bar{c}_t) - v^*] + \phi_{N+1}(\bar{y}_{N+1}) \qquad \text{for} \quad N = 1, 2,\ldots,$$

it is clear that $\lim_{t\to\infty} \phi_t(\bar{y}_t) = 0$. By Lemma 3 we know that $T^{-t}\bar{y}_t \to y^*$ a.s. as $t \to \infty$ ((x^*, y^*, c^*) is the optimal stationary program, hence $\phi_0(y^*) > 0$). Therefore by the monotonicity of ϕ for N sufficiently large

$$\phi_N(\tfrac{1}{2}\bar{y}_N) \geq \phi_N(\tfrac{1}{4} T^N y^*) \geq \phi_0(\tfrac{1}{4} y^*) > -\infty.$$

Hence we proved that $\{\pi_t(\bar{y}_t)\}_{t=0}^{\infty}$ is bounded. This implies that $\{\int \psi_t \bar{y}_t\}_{t=0}^{\infty}$ is bounded.

Remark 1. Since the optimal program satisfies the transversality condition (11) the proof of the last Theorem implies that the sequence of expected values of the prices $\{\int u'(\bar{c}_t)\}$ is bounded.

Remark 2. In [9] it was proved, in the discounted model, that the optimal program satisfies the transversality condition $E\{\delta^t u'(c_t^*)\} \to 0$ as $t \to \infty$. The proof given in [9] works only if $u(0) > -\infty$. In the case $u(0) = -\infty$ (which holds for many important families of utility functions) this assertion can be proved in a similar way as in the above Theorem.

Remark 3. The question of whether the competitive prices for the optimal program are uniformly bounded almost surely arises here. Since it has been shown that $\phi'(x) = u'(g(x))$ holds in the nondiscounted case, the answer to this question is given by the example of Mirman and Zilcha in [10]. This example shows that in the discounted model there exist prices which are not uniformly bounded almost surely. However, the proof in [10] is based on the functional equation (9) and on the validity of the equation $\phi'(x) = u'(g(x))$. Since the valuation function is now defined in the nondiscounted case the same example holds for a model with nondiscounted future utilities. Define

$$\phi(x) = \begin{cases} -e^{1/x} & \text{for } 0 \leq x \leq 1, \\ e \ln x - e & \text{for } x > 1, \end{cases}$$

while $f(x, \eta) = \eta x^{1/2}$ for $0 < \alpha \leq \eta \leq \beta < \infty$. The proof in [10] shows that this valuation function is derived from a strictly concave utility function u, where $\phi'(x) = u'(g(x))$ is satisfied for all $x > 0$. Moreover, if x^* is the optimal stationary capital stock then for any $\epsilon > 0$, Prob $\{x^* \leq \epsilon\} > 0$. This implies that the shadow prices $u'(c^*)$ are not bounded almost surely. This result should be compared to Radner's Theorem 5.1 in [13], where it was proved that $\int u'(c^*) < \infty$, although $u'(c^*)$ might not be bounded. In our case, i.e., in the one-sector model, this can be verified directly from the proof of the above Theorem, using Fatou's Lemma, since $T^{-t}\bar{c}_t \to c^*$ a.s. and $\int u'(\bar{c}_t) \leq \bar{K}$ for all t.

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An Admissible Set Occurring in Various Bargaining Situations*

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Given a set of alternatives S and a binary relation M on S the admissible set of the pair (S, M) is defined to be the set of maximal elements with respect to the transitive closure of M. It is shown that existing solutions in game theory and mathematical economics are special cases of this concept (they are admissible sets of a natural S and M). These include the core of an n-person cooperative game, Nash equilibria of a noncooperative game, and the max-min solution of a two-person zero sum game. The competitive equilibrium prices of a finite exchange economy are always contained in its admissible set. Special general properties of the admissible set are discussed. These include existence, stability, and a stochastic dynamic process which leads to outcomes in the admissible set with high probability.

1. INTRODUCTION AND A DEFINITION

Given a set of alternatives, S, we are interested in finding the possible choices that a society, or a group of individuals, would make from this set. We assume that there is a binary relation, M on S, that describes the valid arguments in the social bargaining process that leads to the final choice. For example, in social choice theory, xMy may mean a majority of the individuals prefer alternative x to alternative y. Let $\hat{M}$ denote the transitive closure of M (i.e., $x\hat{M}y$ if there is a finite sequence of alternatives $x = x_0$, $x_1, x_2, \ldots, x_n = y$ for which $x_{i-1}Mx_i$ for $i = 1, 2, \ldots, n$) then $x\hat{M}y$ means that

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whenever the social bargaining process is at y it may shift to x in a finite number of steps. This approach leads naturally to the following definition.

DEFINITION. The *admissible set of the pair* (S, M) is the set $A(S, M) = \{x \in S \mid y \in S \text{ and } y\hat{M}x \text{ imply } x\hat{M}y\}$.

An equivalent definition of this concept was introduced in a paper by Schwartz [7] and later by Kalai, Pazner, and Schmeidler [3], where they proved the straightforward result:

THEOREM 1. *If S is finite and nonempty then $A(S, M)$ is nonempty.*

One may think of the pair (S, M) as a (bargaining) Markov process where xMy if and only if there is a positive probability of transition from y to x when $x \neq y$. It was shown in [3] that if S is finite then with probability one the bargaining process will enter to and stay in $A(S, M)$ after a finite number of steps. Furthermore, $A(S, M)$ is the smallest subset of S with this property.

The purpose of this work is to study further the properties of the admissible set, and to include the cases where S is infinite. We show that the admissible set is characterized as the set of final outcomes in a qualitative (bargaining) Markov process (Theorem 5). Two general existence theorems are presented; these theorems are general enough to guarantee existence in all the examples that we have considered. We discuss examples where the admissible set is a new solution concept and we show that the admissible set approach leads to a unified treatment of cooperative and noncooperative game theory. More specifically, with the right interpretations, the admissible set coincides with the core of an n-person cooperative game without sidepayments, with the Nash equilibria of a game in the normal form, and it contains the competitive equilibrium prices in the case of an exchange economy with finitely many commodities and traders.

2. EXISTENCE AND CHARACTERIZATION

A straightforward extention of Theorem 1 to the infinite case is

THEOREM 2. *If S is a nonempty compact topological space and if for every $x \in S$ the set $\{y \in S \mid xMy\}$ is open, then $A(S, M)$ is nonempty.*

Proof. If there exists a $y \in S$ such that for no $x \in S$ xMy then $y \in A(S, M)$ and the proof is completed. So assume that this is not the case; then the collection $\{O_x\}_{x \in S}$, where $O_x = \{y \in S \mid xMy\}$ is an open cover of S. Therefore there exists a finite subset of S, $X = \{x_1, x_2, \ldots, x_n\}$, such that $\{O_{x_i}\}_{i=1}^{n}$ is a finite subcover. $A(X, \hat{M})$ is not empty and $A(X, \hat{M}) \subset A(S, M)$. Q.E.D.

When the binary relation is transitive, the corresponding admissible set consists of the maximal elements (i.e., those that are not strictly dominated). The special continuity condition of the next theorem is useful.

THEOREM 3. *Let S be a nonempty compact topological space and let R be a transitive, binary relation such that for every $x \in S$ the set $\{y \in S \mid yRx\}$ is closed; then $A(S, R)$ is nonempty.*

Proof. We have to show that S contains an R-maximal element (i.e., an element $y \in S$ such that for every $x \in S$, xRy implies yRx). By Zorn's lemma it suffices to show that every R chain, T, is bounded. For every $x \in T$ let $Rx = \{z \in S: zRx\}$. If $Rx = \varnothing$ for some x then $x \in A(S, R)$. So we assume that for every $x \in S$, $Rx \neq \varnothing$. If zRx then $Rz \subset Rx$. So by the finite intersection property $\bigcap_{x \in T} Rx \neq \varnothing$ and any element in the intersection is a bound for T. Q.E.D.

An interesting mathematical question is whether the use of Zorn's lemma can be omitted from this proof.

In the case of a finite S (as discussed in [3]) $A(S, M)$ has the property of "outer stability" in the following sense. If $y \notin A(S, M)$ then there is an $x \in A(S, M)$ such that $x\hat{M}y$. This property is lost in the infinite case even when the conditions of Theorem 2 are met. (As an example consider the set of imputations of a symmetric, nonconvex, 0–1 normalized, three-person game with sidepayments and the domination relation.) However, if we deal with an admissible set $A(S, R)$, where R satisfies the conditions of Theorem 3, then the following is true.

THEOREM 4. *If $y \notin A(S, R)$ then there is an $x \in A(S, R)$ such that xRy (provided that R and S satisfy the conditions of Theorem 3).*

Proof. For every $y \in S$, $Ry = \{x \in S \mid xRy\}$ is compact. So $A(Ry, R) \neq \varnothing$ and, by the transitivity of R, $A(Ry, R) \subset A(S, R)$. Q.E.D.

Another property of $A(S, M)$ which is carried over from the finite case to the arbitrary S is that of $A(S, M)$ being the union of the minimally M-closed subsets of S. For an arbitrary (S, M) a subset T of S is said to be *M closed* if it is nonempty and for every $y \notin T$ and every $x \in T$ not yMx. T is *minimally M closed* if it is M closed and no proper subset of T is M closed.

THEOREM 5. *$A(S, M)$ is the union of the minimally M-closed subsets of S.*

Proof. Let T be a minimally M-closed subset of S, $y \in T$ and suppose, per absurdum, that $y \notin A(S, M)$. Then there is an $x \notin T$ such that $x\hat{M}y$ and not $y\hat{M}x$. Hence it follows that the M-closed set $\{x\} \cup \{z \in S \mid z\hat{M}x\}$ is a proper subset of T, which is the desired contradiction.

Conversely, it is shown that if $y \in A(S, M)$ then y belongs to some minimally M-closed subset, specifically the set $T = \{y\} \cup \{x \in S \mid x\hat{M}y\}$. The set T is clearly M closed. If for some $x \neq y$, $x\hat{M}y$ then $y\hat{M}x$ because $y \in A(S, M)$. Hence, any M-closed subset of T contains y and therefore T is minimally M closed. Q.E.D.

The characterization of the admissible set given in Theorem 5 shows that the admissible set is a union of pairwise disjoint sets. The elements of each such set are symmetrically connected by the relation $\hat{M}$ but two elements belonging to two distinct sets are $\hat{M}$ incomparable.

3. APPLICATIONS TO GAMES IN CHARACTERISTIC FUNCTION FORM

EXAMPLE 1. Let V be an n-person cooperative game (without sidepayments). That is $N = \{1, 2,..., n\}$ is the set of players, and for each $E \subset N$, $E \neq \emptyset$, V_E is a closed nonempty subset of R_+^N, the nonnegative orthant of the n-dimensional Euclidean space. It is further assumed that if $0 \leq x \leq_E y \in V_E$, then $x \in V_E$. (Inequalities in R^N are assumed to hold coordinatewise. $\leq_E$ means that the inequality holds only for the coordinates in E.) Finally, V_N is assumed to be compact.

In order to apply Theorem 2, let $S = V_N$ and M be the dominance relation, i.e., xMy if $x \in V_E$ for some $E \subset N$ and $y <_E x$.

COROLLARY 1. *$A(S, M)$ is nonempty.*

Proof. It is easy to check that the domination relation is open (in the sense of Theorem 2) in the relative topology of V_N, thus by Theorem 2 $A(S, M) \neq \emptyset$. Q.E.D.

The next proposition shows that under some weak assumptions on the game the admissible set coincides with the core of the game whenever the latter is not empty.

PROPOSITION 1. *Let V be any game with a nonempty core and in which for every $E \subset N$ V_E contains a point with strictly positive coordinates. Then $A(S, M)$ coincides with the core of V.*

Proof. Recall that the core of V is the set $C(V) = \{x \in V_N \mid$ for every $y \in R^N$ not $yMx\}$. Clearly $C(V) \subset A(S, M)$. Now let $y \in V_N$ such that $y \notin C(V)$. If there exists a $z \in V_N$ such that zMy through a subset E of N for which $E \neq N$ then we proceed in the following way. There exists a positive number ϵ such that $(\epsilon, \epsilon,..., \epsilon) \in (\bigcap_{i=1}^n V_{\{i\}}) \cap V_N$. There exists a $u \in V_N$ such that uMy through E and $u^i < \epsilon$ for every $i \notin E$. There exists a $w \in V_N$ such that wMu through some $\{j\}$ where $j \in \bar{E} = N - E$ and for which $w^i < \epsilon$ for $i = 1,$

$2, \ldots, n$. For every point $x \in C(V)$ we have xMw and thus $x\hat{M}y$. If the M domination on y can be done only through N and wMy then we let $x = y + t(w - y)$, where $t = \max\{r \mid y + r(w - y) \in V_N\}$; thus x is on the boundary of V_N and all of its coordinates are greater than these of y. It follows that $x \in C(V)$ and that xMy. In either case we can find a point $x \in C(V)$ such that $x\hat{M}y$ thus $A(S, M) = C(V)$. Q.E.D.

A probabilistic analogy of this result was presented in [2] and later in [6]. It was shown there that if the transitions between payoffs in V_N are governed by a Markov process compatible with the relation M, then each trajectory converges to the core with probability one.

An interpretation of the relation M, relevant in this context, is that of qualitative probability. More specifically we distinguish only between probable and improbable events. xMy means that the transition from y to x is probable (has a positive quantitative probability) and not xMy means that the probability of transition from y to x is null. Proposition 1 shows that the qualitative approach suffices to obtain the convergence to the core via the dominance relation.

4. APPLICATIONS TO GAMES IN NORMAL FORM

Let S be the set of n lists of strategies in an n-person game in the normal form. $S = \times_{i=1}^{n} S^i$, where S^i is the simplex of mixed strategies of player i. We can define various relations M on S, for the cooperative and noncooperative cases, which yield new and old solution concepts.

EXAMPLE 2. Define xMy if every player i for whom $x^i \neq y^i$ gets a higher payoff at x than at y. The admissible set $A(S, M)$ is a new cooperative solution concept for n-person games in the normal form. (It contains, of course, the strong Nash equilibria.) However, existence of $A(S, M)$ in this case is not implied by either Theorem 2 or 3 and it is still an open question. On the other hand $A(S, M)$ may be very large for some games and may even coincide with S (for example, in the prisoner's dilemma game).

A natural way to extend the relation M is by considering its closure. More precisely, define the relation R on S by xRy if and only if $x \in$ closure $(\{z \in S \mid zMy\} \cup \{y\})$. In this case R satisfies the conditions of Theorem 3 and $A(S, R) \neq \emptyset$. The proof of this fact is similar to the proof of Corollary 2, which follows.

EXAMPLE 3. xMy if for some player j $x^j \neq y^j$, $x^i = y^i$ for every $i \neq j$, and j's payoff at x is higher than his payoff at y. In other words, we restrict the bargaining to be done noncooperatively.

With this definition $A(S, M)$ may still be too large. For example, in the two-person zero sum matching pennies game the admissible set is identical to all of S. This example, provided to us by R. J. Aumann, motivated the following definition of the relation R. Let xRy if and only if $x \in$ closure $(\{z \in S \mid z\hat{M}y\})$. With this new definition, the existence of the admissible set is assured and the admissible set exhibits interesting properties as demonstrated by the following corollary and proposition.

COROLLARY 2. *The relation R fulfills the conditions of Theorem 3 (hence $A(S, R) \neq \varnothing$).*

Proof. The only nontrivial step of the proof is to show that R is transitive. Suppose that xRy and yRz. We want to show that xRz. Let O_x be any neighborhood of x; we will show that the set $\hat{M}^{-1}(O_x) = \{w \in S \mid x'\hat{M}w$ for some $x' \in O_x\}$ is a neighborhood of y. This would mean that every neighborhood of x contains a point where $\hat{M}$ dominates z; thus xRz and the proof would be completed.

It suffices to show that if $u\hat{M}y$ and O_u is any neighborhood of u then $\hat{M}^{-1}(O_u)$ is a neighborhood of y. Since the $\hat{M}$ domination is done by a finite number of M dominations it suffices to show that if uMv and O_u is any neighborhood of u then $M^{-1}(O_u) = \{w \in S \mid u'Mw$ for some $u' \in O_u\}$ is a neighborhood of v. So assume that uMv and O_u is a neighborhood of u and assume further without loss of generality that the domination is done through player 1 (i.e., $u^i = v^i$ for $i \neq 1$ and 1's payoff at u is greater than his payoff at v). It can be easily shown by the continuity of 1's payoff that

$$M^{-1}(O_u \cap \{t \in S \mid t^1 = u^1\})$$

contains a neighborhood of v. Q.E.D.

PROPOSITION 2. *In Example 3, $A(S, R)$ contains the Nash equilibria, and in the case of a two-person zero sum game $A(S, R)$ coincides with the Nash equilibria (minimax strategies).*

Proof. It is obvious that $A(S, R)$ contains the Nash equilibria since for every equilibrium point x there is no $y \neq x$ such that yRx. For the second part of the proposition we assume that the game is a two-person zero sum game and that (x_0, y_0) is any pair of mixed strategies. We will show that there is a pair (x_1, y_1) of minimax strategies such that $(x_1, y_1)\ R(x_0, y_0)$. Let (u, v) be a fixed pair of minimax strategies. If the payoff to player 1 at (x_0, y_0), $V(x_0, y_0)$, equals $V(u, v)$ then either (x_0, y_0) is a minimax pair or one of the players can improve by changing his strategy. This will result in a new point which R dominates (x_0, y_0). So we assume without loss of generality that $V(x_0, y_0) > V(u, v)$. $V(x_0, v) \leq V(u, v)$, and if strict inequality holds then the proof is completed because then $(u, v)\, M(x_0, v)\, M(x_0, y_0)$. So we assume that $V(x_0, v) = V(u, v)$. If (x_0, v) is an equilibrium point then we can stop and if not, then we can find a strategy y, as close as we wish to v, in which $V(x_0, y) < V(u, v)$. Now exchanging the roles of the two players we can either find an equilibrium point (x, y) such that $(x, y)\, R(x_0, y)$, or we find a point (x, y) for which x is as close to u as we wish and $V(x, y) < V(u,v)$. Continuing to exchange the roles of the two players we are either stopped by reaching an equilibrium point (x_1, y_1) which R-dominates (x_0, y_0), or we produce a sequence of points which converges to (u, v), and then $(u,v) R(x_0, y_0)$. In either case we find a minimax point which R-dominates (x_0, y_0). Q.E.D.

A somewhat different relation, L, on the space of mixed strategies, S, yields an admissible set which always coincides with the Nash equilibria. We define L in the following example.

EXAMPLE 4. For a given $x \in S$ we define the set of *possible replies to* x (PR(x)) as follows. Consider subsets of S, B, which satisfy the following three conditions.

(1) $B = \times_{i=1}^{n} B^i$, where $B^i \subset S^i$ (i's mixed strategies).

(2) $x \in B$.

(3) If $y \in B$, $z^i \in S^i$ and player i's payoff at $(y^1, \ldots, y^{i-1}, z^i, y^{i+1}, \ldots, y^n)$ is higher than his payoff at y, then $z^i \in B^i$.

It is easy to check that intersections of sets satisfying (1), (2), and (3) also satisfy these conditions. Also S itself satisfies these three conditions, thus we can and we do define the PR(x) to be the minimal set satisfying (1), (2), and (3).

Intuitively one may think of a possible reply as a list of strategies of the n players, where each player can rationalize (in the sense of (3)) his strategy by considering possible rationalizations of the other players. (A constructive way of defining PR(x), which may be more intuitive, is given in the proof of the next lemma.)

We define the relation L on S by yLx if $y \in$ closure (PR(x)).

PROPOSITION 3. *For every game, $A(S, L)$ consists of precisely the Nash equilibrium strategies. Moreover, for every $x \in S$ there is a Nash equilibrium point y such that $y \in$* PR(x).

Proof. We first observe that a point is an equilibrium point if and only if it is the only possible reply to itself, or equivalently the only point that L dominates it is itself. Thus it suffices to prove the second part of the proposition.

Next we show that if $y \in S$ is an equilibrium point "relative to" $Lx = \{z \in S \mid z\, Lx\}$ (no player can improve his payoff by changing the point y to a point z in Lx) and $y \in Lx$, then y is an equilibrium point relative to all of S

(or just an equilibrium point). If y is an equilibrium point relative to Lx and it is not an equilibrium point, then there is a $z \in (S - Lx)$ and a player j such that $z^i = y^i$ for every $i \neq j$ and j's payoff at z is higher than his payoff at y. But then for every neighborhood of z, O_z, the set $\{w \in S \mid \text{for some } u \in O_z u^i = w^i \text{ for } i \neq j \text{ and } j\text{'s payoff at } u \text{ is higher than at } w\}$ is a neighborhood of y. This implies that $z \in Lx$, which is a contradiction.

Now it suffices to show that Lx contains an equilibrium point relative to Lx. We proceed by the method outlined by Nash [5]. By the lemma that follows Lx is convex and compact. For every $y \in Lx$, let $\Phi(y) = \{z \in Lx \mid z \text{ is a best reply to } y \text{ in } Lx\}$, where z is a best reply to y in Lx if for every player i i's payoff at $(y^i,\ldots, y^{i-1}, \hat{z}^i, y^{i+1},\ldots, y^n)$ is maximized, over Lx, at $\hat{z}^i = z^i$. It follows that for every $y \in Lx$, $\Phi(y)$ is convex, compact, and nonempty. Also $\Phi(y)$ is an upper semicontinuous correspondence, so by Kakutani's fixed point theorem there is a $y \in Lx$ such that $y \in \Phi(y)$. But any point which is a best reply to itself must be an equilibrium point. Q.E.D.

LEMMA 1. *For every* $x \in S$ PR(x) *is convex.*

Proof. We use the following alternative way of defining PR(x):

$$\mathrm{PR}_1(x) = \mathop{\mathsf{X}}_{i=1}^{n} \mathrm{PR}_1^i(x) = \mathop{\mathsf{X}}_{i=1}^{n} \{x^i\} = \{x\}.$$

For $j = 1, 2, 3,\ldots$, for $i = 1, 2,\ldots, n$,

$$\mathrm{PR}_{j+1}^i(x) = \{z^i \in S^i \mid \text{for some } y \in \mathrm{PR}_j(x)\ i\text{'s payoff at } (y^i,\ldots, y^{i-1}, z^i, y^{i+1},\ldots, y^n) \text{ is higher than his payoff at } y\},$$

$$\mathrm{PR}_{j+1}(x) = \mathrm{PR}_j(x) \cup \mathop{\mathsf{X}}_{i=1}^{n} \mathrm{PR}_{j+1}^i(x).$$

It is clear that $\bigcup_{j=1}^{\infty} \mathrm{PR}_j(x)$ is a subset of PR(x) and that it satisfies conditions (1), (2), and (3) in the definition of PR(x), hence $\mathrm{PR}(x) = \bigcup_{j=1}^{\infty} \mathrm{PR}_j(x)$.

To show the convexity of PR(x), it suffices to show the convexity of $\mathrm{PR}^i(x)$ for $i = 1, 2,\ldots, n$ (recall that $\mathrm{PR}(x) = \mathsf{X}_{i=1}^{n} \mathrm{PR}^i(x)$) and therefore it suffices to show:

for $j = 1, 2, 3,\ldots$, and for $i = 1, 2,\ldots n$, $\quad$ Convex Hull $(\mathrm{PR}_j^i(x)) \subset \mathrm{PR}^i(x)$.

We use induction on j. For $j = 1$, the statement is trivial since $\mathrm{PR}_1^i(x) = \{x^i\}$. In order to prove the statement for an integer $j > 1$ we make the following claim.

Claim. If $y \in \mathrm{PR}(x)$, $(y^i + \delta^i) \in S^i$ and i's payoff at $(y^1,\ldots, y^{i-1}, y^i + \delta^i,$

$y^{i+1},\ldots,y^n)$ is higher than his payoff at y then for every $z^i \in \mathrm{PR}^i(x)$ and every $\lambda > 0$ whenever $z^i + \lambda\,\delta^i \in S^i$ $z^i + \lambda\,\delta^i \in \mathrm{PR}^i(x)$.

We leave the proof of this claim to the end and first complete the inductive proof. Assuming that $j > 1$, i is any player, y^i, $z^i \in \mathrm{PR}_j^i(x)$, α, $\beta > 0$, and $\alpha + \beta = 1$ we want to show that $\alpha y^i + \beta z^i \in \mathrm{PR}^i(x)$. There are u and w in $\mathrm{PR}_{j-1}(x)$ such that i's payoff at $(u^1,\ldots,u^{i-1},y^i,u^{i+1},\ldots,u^n)$ is strictly better than his payoff at u and similarly for z^i and w.

$$\alpha y^i + \beta z^i = \alpha u^i + \beta w^i + \alpha(y^i - u^i) + \beta(z^i - w^i).$$

By induction hypothesis $\alpha u^i + \beta w^i \in \mathrm{PR}^i(x)$. Also $\alpha u^i + \beta w^i + \alpha(y^i - u^i) \in S^i$ because it equals $\alpha y^i + \beta w^i$. So by our claim, with δ^i being $y^i - u^i$, λ being α, y being u, and z being $\alpha u^i + \beta w^i$, it follows that $\alpha u^i + \beta w^i + \alpha(y^i - u^i) \in \mathrm{PR}^i(x)$. Now applying the claim again, with δ^i being $z^i - w^i$, y^i being w^i, λ being β, and z^i being $\alpha u^i + \beta w^i + \alpha(y^i - u^i)$ we conclude that $\alpha y^i + \beta z^i \in \mathrm{PR}^i(x)$, which completes the inductive proof.

To prove our claim, we observe that i's payoff at a vector of the type $(y^1,\ldots,y^{i-1},w^i,y^{i+1},\ldots,y^n)$ is $w^i \cdot V(y^1,\ldots,y^{i-1},y^{i+1},\ldots,y^n)$, where V is a vector which does not depend on y^i. Thus the hypothesis in the claim shows that $\delta^i \cdot V(y) > 0$, which implies that $\lambda\,\delta^i \cdot V(y) > 0$ for every $\lambda > 0$. It follows that i's payoff at $(y^1,\ldots,y^{i-1},z^i + \lambda\,\delta^i,y^{i+1},\ldots,y^n)$ is higher than his payoff at $(y^1,\ldots,y^{i-1},z^i,y^{i+1},\ldots,y^n)$, which implies that $z^i + \lambda\,\delta^i \in \mathrm{PR}^i(x)$ and completes the proof of the claim. Q.E.D.

The complexity of the relation L points out that even under assumptions of costless communication among players, the solution concept of Nash equilibrium requires considerable computational ability from the players.

5. CONCLUDING REMARKS

Other solution concepts can be shown to coincide with the admissible set that arises from natural definitions of S and M. These include the α- core and the β- core for games in normal form. For games with a continuum of players (with or without sidepayments) there are several ways of defining S and M so that we are assured of the existence of $A(S, M)$ and so that $A(S, M)$ contains the cores of such games.

Another example which differs from those mentioned or discussed previously is the Walrasian economy.

EXAMPLE 5. Let S be the simplex of normalized price vectors in an exchange economy with l commodities and let $f\colon S \to R^l$ be the excess demand function in this economy. Define pMq if: $p^i > q^i$ if and only if $f^i(q) > 0$ and $p^i < q^i$ if and only if $f^i(q) < 0$. Under standard assumptions on excess demand functions $q^i = 0$ implies that $f^i(q) > 0$, hence M is well defined. The admissible set $A(S, M)$ contains all the Walras equilibria.

Defining the relation R by: xRy iff $x \in$ closure $(\{z \in S \mid z\hat{M}y\})$ leads to a smaller admissible set, $A(S, R)$, which may still contain cycles in addition to Walras equilibria. It seems an interesting problem to find a binary relation of S, derived from M, so that the corresponding admissible set will coincide with the Walras equilibria. (An analogy to Example 4 and Proposition 3 of Section 5.) This approach to stability may prove to be less restrictive than the classical stability analysis.

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The Equivalence of Strong Positive Association and Strategy-proofness

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1. INTRODUCTION

Consider a group that must select one alternative from a set of three or more alternatives. Each member casts a ballot that the voting procedure counts. For a given alternative x, let two ballot profiles C and D have the property that if a member ranks alternative x above alternative y within C, then he also ranks x above that y within D. Strong positive association requires that if the voting procedure selects x when the profile is C, then it must also select x when the profile is D. We prove that strong positive association is equivalent to strategy-proofness. It therefore follows that no voting procedure exists that satisfies strong positive association, nondictatorship, and citizens' sovereignty.

Define a *group* to be a set N whose $|N|$ elements are the group's *members*. They select one element from the *set of alternatives*, S, by each casting a ballot and then using a voting procedure to count the ballots. A *ballot* B_i is a strict ordering of the elements within S, e.g., $B_i = (xyz)$ where $S = \{x, y, z\}$ and x is ranked highest, y second highest, and z lowest. Indifference is not allowed. A *voting procedure* is a single-valued function $v(B_1, ..., B_n)$ that evaluates the profile of ballots and selects one element of S as the group's chosen alternative.

Each member $i \in N$ has preferences P_i over the set of alternatives S. *Preferences*, like a ballot, are a complete, asymmetric, and transitive ordering of S. A member's preferences P_i describe what he truly desires. For example, $P_i = (xyz)$ denotes that individual i most prefers that the group's choice be x, next prefers that it be y, and least prefers that it be z. An alternative notation for the preference ordering $P_i = (xyz)$ is xP_iy, xP_iz, and yP_iz, where xP_iy means individual i prefers x to y. Similarly an alternative notation for the ballot $B_i = (xyz)$ is xB_iy, etc. Beyond completeness, asymmetry, and transitivity we place no restrictions, such as single-peakedness, on either

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admissible preferences or admissible ballots. Any strict ordering is admissible. Indifference, however, is excluded as inadmissible.

A group member's choice of ballot B_i need not be identical to his true preferences P_i. If a member selects a ballot B_i which is identical to his preferences P_i, then $B_i = P_i$ is called his *sincere strategy*. If, however, he selects a ballot B_i which is different than his preferences, then $B_i \neq P_i$ is called his *insincere strategy*. The n-tuple of sincere strategies $B \equiv (B_1, ..., B_n) = (P_1, ..., P_n) = P$ is called the *sincere strategy profile*.

Consider two related ballot profiles $B = (B_1, ..., B_n)$ and $B' = (B_1', ..., B_n')$. Suppose that $v(B_1, ..., B_n) = x$. Suppose also, that for all members $i \in N$ and all alternatives $y \in S$, if alternative x is ranked above alternative y on ballot B_i, then on ballot B_i' alternative x is also ranked above alternative y. In other words, the switch from profile B to profile B' precludes any alternative y, which was ranked below alternative x on ballot B_i, from jumping above alternative x on ballot B_i'. Given these ballot profiles B and B', a reasonable requirement to place on the manner in which v counts the ballots is that $v(B_1', ..., B_n') = x$. After all, $v(B_1, ..., B_n) = x$ and, in the switch from profile B to profile B', alternative x has retained or improved its relative position with respect to every other alternative. We call this requirement, which will be defined more formally shortly, strong positive association.

An example where $S = \{w, x, y, z\}$ and $|N| = 5$ shows that both plurality rule and the Borda count fail to satisfy this requirement. Let profiles B and B' be:

$$\begin{aligned}
B_1 &= (xwzy), & B_1' &= (xyzw),\\
B_2 &= (xwzy), & B_2' &= (xzyw),\\
B_3 &= (yzxw), & B_3' &= (yzxw),\\
B_4 &= (zyxw), & B_4' &= (yzxw),\\
B_5 &= (wyxz), & B_5' &= (yxwz).
\end{aligned}$$

With respect to alternative x, profiles B and B' satisfy the requirements of strong positive association. Note that the condition's requirements on B and B' relate only to those pairs of alternatives that include x, not to all possible pairs of alternatives. Consider plurality rule (v_p) first. Profile B gives x two first place votes, compared to one first place vote each for the other elements of S. Therefore $v_p(B) = x$. But contrary to the requirement of strong positive association, $v_p(B') = y$. The story repeats itself for the Borda count (v_B).[1] For profile B alternative x receives 9 points while the other alternatives each receive 7 points. For profile B', alternative y receives 12 points while alternatives w, x, and z, respectively, receive 1, 10 and 7 points. Therefore $v_B(B) = x$ and $v_B(B') = y$.

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[1] The Borda count, which is named after its eighteenth century French inventor, selects a winning alternative by assigning each alternative $(|S| - k - 1)$ points for each ballot in which it is ranked k positions from the top. The points for each alternative are summed and the winner is that alternative which receives the most points. If two alternatives receive the same number of points, then individual one's ballot, B_1, is used to break the tie.

2. EQUIVALENCE OF STRONG POSITIVE ASSOCIATION AND STRATEGY-PROOFNESS

The formal definitions of the two conditions are:

Strong positive association (SPA).[2] For any $x \in S$, let C and D be any two ballot profiles such that, for all $s \in S$ and all $i \in N$, xC_is implies xD_is. A voting procedure satisfies SPA if and only if $v(C) = x$ implies $v(D) = x$.

Strategy-proofness (SP). A voting procedure satisfies SP if and only if no sincere strategy profile $C = P$ exists such that, for some member $i \in N$ and for some insincere strategy C_i',

$$v(C/C_i')\, P_i\, v(C/C_i), \tag{1}$$

where

$$C/C_i' = (C_1, \dots, C_{i-1}, C_i', C_{i+1}, \dots, C_n) = P/C_i' \tag{2}$$

and

$$C/C_i = (C_1, \dots, C_{i-1}, C_i, C_{i+1}, \dots, C_n) = P/P_i = P. \tag{3}$$

Strategy-proofness requires that no member has an incentive to employ an insincere strategy. Specifically, notice in (1) that $C_i = P_i$ because C_i is the sincere strategy of member i; therefore (1) may be rewritten as $v(C/C_i')\, C_i\, v(C/C_i)$.

THEOREM. *A voting procedure satisfies* SP *if and only if it satisfies* SPA.[3]

Proof. First we prove that SP implies SPA. Suppose that SP does not imply SPA. Therefore a voting procedure v exists that satisfies SP, but not

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[2] We have named this condition "strong positive association" because it is a straightforward strengthening of Arrow's condition of "positive association" (see [1, p. 26]). One can show that SPA implies positive association. In addition, one can show that if, in our definition of a voting procedure, we permitted the set S of feasible alternatives to vary over some universal set of alternatives, then SPA also implies Arrow's independence of irrelevant alternatives condition. The converse, however, is not true. For further discussion of the relationship between SP (and SPA) and Arrow's condition see [2] and [6].

[3] If we had not assumed that every strict ordering over S is admissible as members' preferences and ballots, then the theorem would hold in one direction only. Inspection of the proof shows that if the set of admissible preferences and ballots is restricted, then SP implies SPA while SPA does not necessarily imply SP.

SPA. This means that distinct alternatives x, $z \in S$ and profiles B and C must exist such that, for all $i \in N$ and all $y \in S$,

$$xB_iy \rightarrow xC_iy, \tag{4}$$

$v(B) = x$, and $v(C) = z$. Consider the sequence

$$\begin{aligned}
&v(B_1, B_2, \dots, B_n) = x,\\
&v(C_1, B_2, \dots, B_n),\\
&\dots\\
&v(C_1, \dots, C_{j-1}, B_j, B_{j+1}, \dots, B_n) = v(D/B_j),\\
&v(C_1, \dots, C_{j-1}, C_j, B_{j+1}, \dots, B_n) = v(D/C_j),\\
&\dots\\
&v(C_1, \dots, C_{n-1}, B_n),\\
&v(C_1, \dots, C_{n-1}, C_n) = z.
\end{aligned} \tag{5}$$

Since $v(B) = x$ and $v(C) = z$, a switching point must exist; a $j \in N$ exists such that $v(D/B_j) = x$ and $v(D/C_j) = u$ (u might equal z). Two possibilities exist for B_j, either xB_ju or uB_jx, and both lead to contradictions. If xB_ju, then (4) implies that xC_ju. Therefore $v(D/B_j)\,C_j\,v(D/C_j)$, which is a contradiction of SP. If uB_jx, then $v(D/C_j)\,B_j\,v(D/B_j)$, which is also a contradiction of SP. Therefore SP implies SPA.

We now show that SPA implies SP. Suppose that SPA does not imply SP. Therefore a voting procedure v must exist that satisfies SPA but not SP. Consequently a distinct pair $x, y \in S$, a sincere strategy $B_j = P_j$, a profile B, a member $j \in N$, and an insincere strategy B_j' must exist such that

$$v(B/B_j') = y, \tag{6}$$

$$v(B/B_j) = x, \tag{7}$$

and

$$yB_jx. \tag{8}$$

Partition S into three exhaustive and disjoint subsets:

$$W^+ = \{z \in S \mid zB_jx\}; \tag{9}$$

$$X = \{z \in S \mid (xB_jz \ \&\ zB_j'x) \text{ or } z = x\}; \tag{10}$$

$$W^- = \{z \in S \mid xB_jz \ \&\ xB_j'z\}. \tag{11}$$

Construct a ballot Q_j from ballots B_j and B_j' as follows:

$$[s \in W^+ \ \& \ (t \in X \text{ or } t \in W^-)] \Rightarrow sQ_jt; \tag{12}$$

$$(s \in X \ \& \ t \in W^-) \Rightarrow sQ_jt; \tag{13}$$

$$s, t \in W^+ \Rightarrow (sQ_jt \Leftrightarrow sB_jt); \tag{14}$$

$$s, t \in X \Rightarrow (sQ_jt \Leftrightarrow sB_j't); \tag{15}$$

and

$$s, t \in W^- \Rightarrow (sQ_jt \Leftrightarrow sB_jt). \tag{16}$$

The effect of this construction is to order the three sets W^+, X, and W^- in descending order: $Q_j = (W^+XW^-)$. The individual alternatives within W^+ are ordered as ballot B_j ordered them, the alternatives within X are ordered as ballot B_j' ordered them, and the alternatives within W^- are ordered as ballot B_j ordered them. Notice that (10) and (14) imply that if $s \in X$ and $s \neq x$, then sQ_jx.

Denote by w the group choice that the ballot profile B/Q_j generates: $v(B/Q_j) = w$. Since W^+, X, and W^- partition S, three possibilities exist for w: $w \in W^+$, $w \in X$, or $w \in X^-$. We consider each of these possibilities in turn and show that application of SPA, which v is assumed to satisfy, leads to a contradiction of our assumptions (6), (7), or (8). Therefore SPA implies SP.

If $w \in W^+$, then for all $z \in S$, (12) and (14) imply that

$$wQ_jz \Rightarrow wB_jz. \tag{17}$$

Moreover, $v(B/Q_j) = w$, and the only difference between ballot profile B/Q_j and B/B_j is the ballot of member j. Therefore SPA is applicable to the switch from profile B/Q_j to B/B_j. It implies that $v(B/B_j) = w$ because $v(B/Q_j) = w$. But our assumption (7) states that $v(B/B_j) = x$. Since $x \notin W^+$ we have our first contradiction.

If $w \in X$, then for all $z \in S$, (10), (11), (13), and (15) imply that

$$wQ_jz \Rightarrow wB_j'z. \tag{18}$$

The same argument, *mutatis mutandis*, as we used in analyzing the first possibility implies that SPA is applicable to the switch from profile B/Q_j to profile B/B_j'. Therefore, $v(B/B_j') = w$ because $v(B/Q_j) = w$. Assumption (6), however, is that $v(B/B_j') = y$. Moreover, (6) when coupled with (9) implies that $y \in W^+$. Therefore, a contradiction exists because $w \in X$ and y equals w.

The one remaining possibility is $w \in W^-$. If $w \in W^-$, then for all $z \in S$, (12), (13), and (16) imply that

$$wQ_jz \Rightarrow wB_jz. \tag{19}$$

The same argument, *mutatis mutandis*, implies that SPA is applicable to the switch from profile B/Q_j to profile B/B_j. Therefore $v(B/B_j)$ w because $v(B/Q_j) = w$. But assumption (7) states that $v(B/B_j)$ x. Since $x \notin W'$ we have our third contradiction. ▌

A corollary to the equivalence theorem is that if the number of alternatives is at least three, then no voting procedure exists that satisfies SPA, citizens' sovereignty, and nondictatorship. Citizens' sovereignty requires that the group can actually choose any alternative within S by casting an appropriate ballot profile. A dictatorial voting procedure vests all decision making power into one individual, the dictator. Formally these two conditions and the corollary are:

Citizens' sovereignty (CS). A voting procedure satisfies CS if and only if, for every alternative $x \in S$, a ballot profile B exists such $v(B)$ x.

Dictatoriality (D). A voting procedure is dictatorial if and only if a member $i \in N$ exists such that, for all ballot profiles B and all alternatives $x \in S$, either $v(B) = x$ or $v(B)\, B_i x$.

COROLLARY. *If S contains at least three alternatives, then every voting procedure that satisfies* SPA *and* CS *is dictatorial.*

The corollary's proof follows directly from the equivalence between SPA and SP and a theorem of Gibbard [3] and Satterthwaite [6]: If S contains at least three alternatives, then every voting procedure that satisfies SP and CS is dictatorial.[4]

3. DISCUSSION

Pattanaik [5] in a recent paper examined the relationship between strategy-proofness and conditions of the "independence of irrelevant alternatives" type. He assumed that voting procedures may select as the group's choice a lottery among several alternatives. This contrasts with our assumption that a voting procedure deterministically selects as the group's choice a single alternative. Within his framework Pattanaik defined several independence conditions, the strongest of which is IC1. He then showed that IC1 is a necessary, but not sufficient, condition for SP. Therefore from our equivalence theorem between SP and SPA it follows that within the context of deterministic voting procedures SPA implies IC1, but not the converse.

Gibbard [4] has also considered, in a somewhat different manner than

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[4] It is easily shown that, in conjunction with SP (or SPA), CS and Pareto optimality are equivalent. Therefore the corollary may be stated with Pareto optimality replacing CS.

Pattanaik, voting procedures that admit lotteries as outcomes. He showed [4, Lemma 2] that such voting procedures are SP if and only if they are "localized" and "nonperverse." This result, together with our equivalence result for SP and SPA, imply that, for the case of deterministic voting procedures, his two conditions of localization and nonperversity are jointly equivalent to our single condition of SPA.

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Competitive and Value Allocations of Large Exchange Economies*

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INTRODUCTION

Consider an exchange economy whose traders have concave utility functions. It is to be understood, in the present paper, that utility has cardinal significance, so we refer to such economies as being cardinal. As an extension of game theoretic concepts, Shapley [21] proposed the following notion of value allocation for a cardinal economy. Let there be n traders and let us say that the positive numbers $\lambda_1, \ldots, \lambda_n$ are equilibrium weights if there is a reallocation of goods to traders such that the weighted utility that every trader obtains is precisely the value [20] of the game where the worth of a coalition is simply the maximum utility that the coalition can get when the utility of trader i is given weight λ_i. The allocation of goods for which $\lambda_1, \ldots, \lambda_n$ is an equilibrium is then called a value allocation. Under standard conditions value allocations exist.

Recent work by, among others, Shapley and Shubik [23], Champsaur [9], and Aumann [3], has established that, as it is the case with the concept of the core (see [19] for an extensive treatment), the value notion provides yet another cooperative game theoretic foundation for the idea of perfect competition. Thus, building on the remarkable theory of values of nonatomic games developed by Aumann and Shapley [4], Aumann [3] has proved under, among others, smoothness hypotheses, that in cardinal economies with a continuum of traders the set of competitive allocations and the set of value allocations coincide.

The purpose of this paper is to give an "asymptotic" version of Aumann's theorem. As in Hildenbrand [19], by asymptotic it is meant that instead of dealing with economies with a continuum of traders, we deal with sequences of finite, but increasingly large, ones. Thanks to the pioneering investigation by Champsaur [9] of value allocations of replica economies, it is well known that if we consider a sequence of cardinal increasingly large replica economies

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limit, then, up to a few technical assumptions, value allocations satisfying a symmetry requirement are eventually close to competitive allocations. Unlike Aumann, Champsaur did not impose in his model smoothness hypotheses. Without them it may happen that competitive allocations are not eventually close to value allocations, and so it is not possible to get a counterpart to Aumann's theorem even for the replica case. Since this is the aim here we shall place ourselves in the same smoothness framework as Aumann's. This restriction in scope is, we feel, fully justified by the stronger results that can be hoped for. For a general nondifferentiable analysis of cardinal economies with a continuum of traders see Hart [17].

Suppose then that we have a sequence of smooth, increasingly large cardinal economies. We will assume that the sequence converges to a definite limit in a sense already used by Hildenbrand. We wish to find conditions for the set of value and competitive allocations to be eventually close. Since, by Aumann's result, they are identical at the limit, it is patent that, informally speaking, all we need is that the limit be a continuity point of both the competitive equilibrium correspondence and the value allocation correspondence. It is well known that a sufficient condition for continuity at the limit of the equilibrium correspondence is that the "limit economy" be regular in the sense introduced by Debreu [10] and generalized to the continuum of traders context by, among others, Dierker [13]. It turns out that if the limit is regular, then it is also a continuity point of the correspondence which assigns to economies the value allocations satisfying a weak symmetry requirement. Thus, by requiring that the given sequence converge to a regular limit we obtain a very general form of the desired result. The regularity condition is easily describable (i.e., an economy is regular if the Jacobian of the excess demand map has full rank at every equilibrium price vector) and, most important, it is generic, i.e., in an appropriate sense almost all limits are regular (see [10, 13]).

Except for some special arguments at the very end, where regularity is exploited, our line of proof is entirely traditional. As a side product to the main result we investigate, by being very explicit on the approximating bounds, which rate of convergence does the proof yield. After all, to quote Aumann and Shapley [4, p. 210], "the asymptotic results imply a framework within which the manner and rate of convergence can be discussed. The continuous formulation, by its nature, precludes such considerations." We get a convergence of the order of $1/n^{1/3}$; it is not claimed that it is the best but we have no suggestion on how to improve it (Shapley and Shapiro [22] also got $1/n^{1/3}$ for the convergence of the value in some weighted majority games). Rate of convergence results for the case of the core have been obtained by Debreu [11] and Grodal [15]. It was Debreu's paper [11] that made very clear the usefulness of the concept of regular limit for the purposes of establishing strong asymptotic theorems.

Section 1 describes the model and assumptions, states the results, discusses open problems, and gives more precise and exhaustive references. Section 2 contains the proofs.

1. THE MODEL

The consumption set is $P = \{x \in R^l : x \gg 0\}$. In the present cardinal utility context a consumer is a pair (u, ω), where $\omega \in P$ is the initial endowment vector and $u: P \to R$ the utility function. We assume:

(1) $u: P \to R$ is a bounded above and below nonnegative C^2 function with $Du(x) \gg 0$ and $D^2u(x)$ negative definite for every $x \in P$ (hence u is concave). Moreover, if $x_n \to x \in \partial P$, then $\| Du(x_n)\| \to \infty$.

Two utility functions u, u' are equivalent if $u = au' + b$, $a > 0$. Therefore, for the sake of definiteness and without loss of generality, we will normalize and assume $u(e) = 1$, $u(2e) = 2$, $e = (1, \ldots, 1) \in P$.

Let $\mathcal{U}$ be the space of (normalized) utility functions satisfying (1). We endow $\mathcal{U}$ with the topology of C^2 uniform convergence on compacta plus C^0 *uniform* convergence. This is a strong topology; our results are valid (without modification of proofs) for the topology of uniform convergence on compacta, but then our assumptions (yet to be made) do not seem to be independent of the normalization chosen, a situation we want to avoid.

The space of consumer's characteristics is $\mathcal{A} = \mathcal{U} \times P$; we denote $a = (u_a, \omega_a)$; $\mathcal{A}$, with the product topology, is metrizable.

As usual, we define the demand correspondence $\varphi: \mathcal{A} \times P \times R_{++} \to P$ by $\varphi(a, p, w) = \{w \in P: px \leqslant w$, and $py \leqslant w$ implies $u_a(x) \geqslant u_a(y)\}$; (1) guarantees that p is U.H.C. and nonempty valued.

A continuum economy is a measure ν on $\mathcal{U} \times P$ (endowed with the product Borel σ field) with compact support.

Let $S = \{p \in P: \| p \| = 1\}$. Given the continuum economy ν, the aggregate excess demand function $\Phi_\nu : S \to R^l$ is given by $\Phi_\nu(p) = \int_{\mathcal{A}} (\varphi(a, p, p\omega_a) - \omega_a)\, d\nu(a)$. As in Hildenbrand [19, p. 119] one verifies the boundary condition: $p_n \to p \in \partial S$ implies $\| \Phi_\nu(p_n)\| \to \infty$. Hence the set $\{p \in S: \Phi_\nu(p) = 0\}$ is compact.

As in Dierker [13], it is verified that the functions $(p, w) \mapsto \varphi(a, p, w)$ and $p \mapsto \Phi_\nu(p)$ are C^1.

Let $\Pi(\nu) = \{p \in S: \Phi_\nu(p) = 0\}$. If $p \in \Pi(\nu)$, the linear map $D\Phi_\nu(p)$ maps $T_p(S)$, the tangent plane to S at $p \in S$, into itself. If $D\Phi_\nu(p)$ does in fact map onto, i.e., if it has the largest possible rank $(l - 1)$, then it is said that Φ_ν is regular at p (see Dierker [13] and her references).

DEFINITION. ν is *regular* if rank $D\Phi_\nu(p) = l - 1$ whenever $\Phi_\nu(p) = 0$.

As a corollary of Dierker's [13] results, it is easy to see that, with a natural topology on the ν's (which will be specified later on) there is an open, dense set of regular economies.

A *finite economy* is a mapping $\mathscr{E}: I \to \mathscr{A}$, where I is a finite indexing set. Two finite economies are equivalent if one can be obtained from the other via an automorphism of I. Hence, up to equivalence, a finite economy is completely specified by its cardinality, i.e., $\#(I)$, and the measure $\nu_{\mathscr{E}}$ on $\mathscr{A}$ defined by $\nu_{\mathscr{E}}(U) = \#\{i \in I: \mathscr{E}(i) \in U\}/\#(I)$. We will always write expressions like $u_{\mathscr{E}(i)}$ as u_i.

DEFINITION. The sequence of finite economies $\mathscr{E}_n$ *converges* to the continuum economy ν (written $\mathscr{E}_n \to \nu$) if: (i) $\#(I_n) \to \infty$, where $\mathscr{E}_n : I_n \to \mathscr{A}$ (ii) the measure $\nu_{\mathscr{E}}$ converges weakly to the measure ν; and (iii) $\operatorname{supp}(\nu_{\mathscr{E}})$ converges to $\operatorname{supp}(\nu)$ with respect to the Hausdorff distance derived from any metric on $\mathscr{A}$.

Given a (finite) economy $\mathscr{E}: I \to \mathscr{A}$ an *allocation* is a map $x: I \to P$; x is *feasible* if $\sum_{i \in I} x_i = \sum_{i \in I} \omega_i$; x is *competitive* if it is feasible and, for some $p \in S$, $x_i \in \varphi(a_i, p, p\omega_i)$ for every $i \in I$; p is then an equilibrium price system. We let $\mathscr{W}(\mathscr{E})$, $\Pi(\mathscr{E})$ be, respectively, the set of competitive allocations and price systems for $\mathscr{E}$. The sets $\mathscr{W}(\mathscr{E})$, $\Pi(\mathscr{E})$ are compact.

For a function $f: I \to R$, $\#(I) < \infty$, denote $|f| = \max_{i \in I} |f(i)|$. By abuse of language, if $x: I \to P$, we let $|x| = \max_{i \in I} \|x_i\|$, where $\| \ \|$ is the euclidean norm. Of course, $| \ |$ is a norm on the set of function from I to P (or from I to R). With respect to the metric of this norm we let d be the Hausdorff distance on the compact subsets of $\{x: I \to P\} = P^I$. More precisely, if $A, B \subset P$. are compact, let $\rho(A, B) = \min\{\epsilon \geq 0$: for every $x \in A$, $|x - y| \leq \epsilon$ for some $y \in B\}$. Put then $d(A, B) = \max\{\rho(A, B), \rho(B, A)\}$. Remember, we always assume $\#(I) < \infty$.

Following Shapley and Shubik [23] and Aumann [2], we can define, for every finite economy $\mathscr{E}: I \to \mathscr{A}$ and $\lambda: I \to R^I_{++}$ (a vector of "weights"), a game in characteristic form by letting the worth of coalition $\phi \neq C \subset I$ be $V(\mathscr{E}, \lambda, C) = \max\{\sum_{i \in I} \lambda_i u_i(x_i): x_i \in P, \sum_{i \in I} x_i \leq \sum_{i \in I} \omega_i\}$; it is easily verified that the maximum exists. Let, also, $V(\mathscr{E}, \lambda, \phi) = 0$. Note that V is homogeneous of degree 1 on λ, i.e., $V(\mathscr{E}, \alpha\lambda, C) = \alpha V(\mathscr{E}, \lambda, C)$, $\alpha \geq 0$.

The game $V(\mathscr{E}, \lambda, \cdot)$ has a Shapley value; let it be $v(\mathscr{E}, \lambda): I \to R$. The value can be computed as follows. For every $i \in I$ let $\mathscr{C}_i$ be the set of coalitions from $I \backslash \{i\}$, player i has a probability distribution π_i on $\mathscr{C}_i$; π_i is uniquely determined from two principles: (i) if $\#(C) = \#(C')$, then $\pi_i(\{C\}) = \pi_i(\{C'\})$; (ii) $\pi_i(\{C \in \mathscr{C}_i : \#(C) = m\})$ is independent of $0 \leq m < \#(I)$. Then $v_i(\mathscr{E}, \lambda) = \sum_{C \in \mathscr{C}} \pi(C)[V(\mathscr{E}, \lambda, C \cup \{i\}) - V(\mathscr{E}, \lambda, C)]$, i.e., the value is an average of marginal contributions. It can be seen that the value *is* an imputation, i.e., $\sum_{i \in I} v_i(\mathscr{E}, I) = V(\mathscr{E}, \lambda, I)$.

DEFINITION. A feasible allocation $x: I \to P$ for the finite economy $\mathscr{E}: I \to \mathscr{A}$ is a (Aumann–Shapley) *value allocation* if for some $\lambda: I \to R_{++}$,

$\lambda_i u_i(x_i) = v_i(\mathscr{E}, \lambda)$ for all $i \in I$. The set of value allocations will be denoted $\mathscr{V}(\mathscr{E})$.

For the purposes of the theorem we feel the need to consider only some subsets of $\mathscr{V}$ satisfying a minimal symmetry (i.e., equal treatment) requirement.

DEFINITION. Given $r \geq 0$, an allocation $x: I \to P$ for the finite economy $\mathscr{E}: I \to \mathscr{A}$ is an *r-value allocation* if for some $\lambda: I \to R_{++}$, $\lambda_i u_i(x_i) = v_i(\mathscr{E}, \lambda)$ *and* $\lambda_i \geq r \mid \lambda \mid$ for all $i \in I$.

The set of r-value allocations will be denoted $\mathscr{V}_r(\mathscr{E})$. So, $\mathscr{V}_r(\mathscr{E}) \subset \mathscr{V}_0(\mathscr{E}) = \mathscr{V}(\mathscr{E})$. The intended interpretation is that r is positive but "very small."
We have:

THEOREM. *If ν is regular and $\mathscr{E}_n : I_n \to \mathscr{A}$, $\# I_n = n$, is a sequence of finite economies converging to ν, then, for some $r > 0$ sufficiently small,*

$$d(\mathscr{V}_r(\mathscr{E}_n), \mathscr{W}(\mathscr{E}_n)) = O(1/n^{1/3}).$$

Presumably, the theorem can be improved in at least three directions.

1. We do not know if (always assuming ν regular) $d(\mathscr{V}(\mathscr{E}_n), \mathscr{W}(\mathscr{E}_n)) \to 0$. It is true that eventually every competitive allocation is close to a value allocation, but is eventually every value allocation close to a competitive allocation? The key unsettled question is: "Does there exist a compact set $Q \subset P$ such that for every $x \in \mathscr{V}(\mathscr{E}_n)$ and $i \in I_n$, $x_i \in Q$?" The analogous statement for the core holds true (a fact first proved by Bewley [2]) and it plays a fundamental role in the asymptotic theory for the core. For every $r > 0$, r-value allocations have this compactness property; if we could rule out sequences of value allocations "escaping to infinity," then the theorem would hold in all generality.

If we had reduced ourselves to a type-economy universe, i.e., if we had assumed that, for all n, $\text{supp}(\nu_{\mathscr{E}_n}) \subset K \subset \mathscr{A}$, where K is finite, then we could have appealed to Champsaur's [9] concept of symmetric value allocations: A value allocation is symmetric if agents with the same characteristics receive the same weight (i.e., $a_i = a_{i'}$ implies $\lambda_i = \lambda_{i'}$). It is easy to check that for symmetric value allocations the compactness property of the last paragraph holds; trivial modifications of the proof yield then the validity of the theorem for that case; more precisely: *if* $\text{supp}(\nu_{\mathscr{E}_n}) \subset K$ *for all* n, K *is finite, and* $\mathscr{V}^*(\mathscr{E}_n)$ *denote the symmetric value allocations, then* $d(\mathscr{V}^*(\mathscr{E}_n), \mathscr{W}(\mathscr{E}_n)) = O(1/n^{1/3})$.

The r-value allocations are the ones that satisfy a weak symmetry requirement: Since the agents with characteristics in $\bigcup_n (\text{supp}(\nu_{\mathscr{E}_n})) \cup \text{supp}(\nu)$ (a compact set) are not "too dissimilar" we prevent the possibility of the ratio between maximal and minimal weights becoming unbounded. It is a legitimate and delicate question to determine if the definition of $\mathscr{V}_r$, or, more properly, the validity of the theorem is independent of the particular normalization used for the utility functions. The answer is yes (i.e., the theorem, as it stands, remains valid) up to a point: We are not free to choose arbitrary representatives of the (dimension 2) utility functions equivalence classes; the assignment $a \mapsto u_a$ has to be continuous. Again, in the type-economy universe this is automatically satisfied.

Strictly speaking, we have not shown that the requirement that ν be regular is needed to get $d(\mathscr{V}_r(\mathscr{E}_n), \mathscr{W}(\mathscr{E}_n)) \to 0$, but there can be no doubt that without it a counterexample can be developed. The idea that the notion of regular continuum economies is the key concept for obtaining strong limit theorems (i.e., continuity in the limit theorems) is Debreu's [11].

2. Our method of proof, which is essentially the same as every proof of the value equivalence theorem given so far [2, 3, 9, 24], yields an order of convergence of $1/n^{1/3}$. This is to be contrasted with the situation for the core, where a rate of convergence of $1/n$ has been proved [11, 15] and with the Shapley and Shubik [23] presumption derived from examples, that a rate of $1/n$ may also be the typical case with the value in *replica* economies. See, however, Shapley and Shapiro [22, 23]. In a nutshell the source of the $\frac{1}{3}$ exponent is the following. The fastest way to bring an expression like $\epsilon(n) + (1/(n\epsilon(n)^2))$ to zero is to put $\epsilon(n) = 1/n^{1/3}$.

3. It is a fact that if ν is regular and $\mathscr{E}_n \to \nu$, then, eventually, $\#\mathscr{W}(\mathscr{E}_n)$ is a constant. It may be conjectured that, eventually, $\#\mathscr{V}_r(\mathscr{E}_n) = \#\mathscr{W}(\mathscr{E}_n)$ provided, of course, that r be small enough.

Our whole setup of looking at economies as distributions and concentrating on sequence of finite economies is due to Hildenbrand [19]; the consideration of large economies (in one form or another) for equivalence theorem purposes goes back to Debreu and Scarf [12], Vind [26], and Aumann [2]. The notion of regular, smooth economies is due to Debreu [10]; the appropriate definitions for the present continuum case have been given by Dierker [13]. Debreu and Smale [25] have results on an alternative characterization of regularity which is also implicit, and basic, in our analysis (Proposition 5). The concept of the Shapley value is, of course, Shapley's [20]. For the case of markets with transferable utility a value equivalence theorem is stated and proved in Shapley and Shubik [23]. Shapley [21] and, implicitly, Harsanyi (see [16]) proposed the extension of the value concept to non-side-payment games (and, in particular, markets) through the use of λ weights. So extended, and using the replica-economy device, the value of large, finite economies has been studied by Champsaur [9] without smoothness hypotheses, in which case competitive allocations may not be close to value allocations. In the continuum situation, and with smoothness assumptions, the equality of competitive and value allocations has been proved by Aumann and Shapley [4] (for transferable utility markets) and Aumann [3](general case) as an application of the very general Aumann-Shapley theory [4] of values of games with a continuum of players. To the extent that this is possible, Hart [17] has generalized Aumann's results in [3] to the nonsmooth case. Recently, Brown and Loeb [8] have given a nonstandard analytic treatment of Aumann's theorem.

For the present paper, Champsaur [9] and Aumann's [3] articles (and, from a different angle, Debreu [11] and Grodal [15]) have been fundamental. Our assumptions on utility functions and on single economies are very close to Aumann's. Champsaur perceived that to get asymptotic theorems for the value a certain well-known family of probability inequalities has to be appealed to. The general outline of the proof (of every proof) seems to go back to Shapley and Shubik [23], which contains also the first published version of the value equivalence theorem.

As a final remark, our boundary assumption on utility functions (1) is certainly impalatable. It makes things easier and isolates the essential points. It should be possible to give a substantially identical version of the theorem with much more acceptable hypotheses; strictly speaking, however, this is yet another open problem.

2. PROOF OF THE THEOREM

2.1. *Some definitions and Preliminaries*

We observe first that because of our boundary assumption (1) on utility functions there is for every $a \in \mathscr{A}$, $\lambda > 0$, and $q \in P$ a unique $x_{a,\lambda}(q) \in P$ such that $\lambda Du_a(x_{a,\lambda}(q)) = q$. Moreover, D^2u_a being nonsingular, the function $q \mapsto x_{a,\lambda}(q)$ is C^1. Note that $x_{a,\lambda}(q) = \varphi(a, q, qx_{a,\lambda}(q))$.

Let $\mathscr{E}_n : I_n \to \mathscr{A}$ and ν satisfy the conditions of the theorem; in particular, $\mathscr{E}_n \to \nu$ and ν is regular. From now on the sequence $\mathscr{E}_n$ and ν are kept fixed, but whenever some property is asserted of the sequence $\mathscr{E}_n$, it should always be understood that the property holds eventually, i.e., for all but possibly finitely many n. We assume $\#(I_n) = n$ and let $I_n = \{1, \ldots, n\}$; $\mathscr{E}$ will denote a generic term of the sequence. If $\mathscr{E}: I \to \mathscr{A}$, $x: I \to P$, $\lambda: I \to R_{++}$, then a_i, u_i, ω_i, x_i, λ_i stand for $\mathscr{E}(i)$, $u_{\mathscr{E}(i)}$, $\omega_{\mathscr{E}(i)}$, $x(i)$, $\lambda(i)$; this notational convention will be used freely (thus x_{a_i,λ_i} becomes x_i, etc.).

As it is seen from examining the corresponding Lagrangean expression, the problem "Max $\sum_{i\in C} \lambda_i u_i(y_i)$ s.t. $y_i \in P$ and $\sum_{i\in C} y_i \leq z$" is solved by $y_i = x_i(q(\mathscr{E}, \lambda, C, z))$, where $q(\mathscr{E}, \lambda, C, z) \in P$ is uniquely determined by $\sum_{i\in C} x_i(q(\mathscr{E}, \lambda, C, z)) = z$. Moreover, $\sum_{i\in C} Dx_i(q) = \sum_{i\in C} [\lambda_i D^2u_i(x_i)]^{-1}$ is a negative definite, hence nonsingular, matrix, and, therefore, $z \to q(\mathscr{E}, \lambda, C, z)$ is C^1. Let $q(\mathscr{E}, \lambda, C, \sum_{i\in C} \omega_i) = q(\mathscr{E}, \lambda, C)$, $q(\mathscr{E}, \lambda, I) = q(\mathscr{E}, \lambda)$. Note that, for $\alpha > 0$, $q(\mathscr{E}, \alpha\lambda, C) = \alpha q(\mathscr{E}, \lambda, C)$. Define $V(\mathscr{E}, \lambda, C, z) = \text{Max}\{\sum_{i\in C} \lambda_i u_i(y_i)$:

$y_i \in P$, $\sum_{i \in C} y_i \leq z\}$. Of course, $V(\mathscr{E}, \lambda, C) = V(\mathscr{E}, \lambda, C, \sum_{i \in C} \omega_i)$. For fixed $\mathscr{E}$, λ, C the function $z \mapsto V(\mathscr{E}, \lambda, C, z)$ is C^2, concave, and satisfies $DV(\mathscr{E}, \lambda, C, z) = q(\mathscr{E}, \lambda, C, z)$.

The set $\mathscr{A}^* = \bigcup_n \operatorname{supp}(\nu_{\mathscr{E}_n}) \cup \operatorname{supp}(\nu)$ is compact. Therefore, given our assumption, there is $M > 0$ such that for all n, $i \in I_n$ and $x \in P$ we have $u_i(x) \leq M$.

2.2. *Lemmata*

LEMMA 1. *For any $s, s' > 0$ the set $Q = \{x \in P: x \leq se, \| Du_a(x) \| \leq s'$ for some $a \in \mathscr{A}^*\}$ is compact.*

Proof. Let $a \in \mathscr{A}^*$. By (1) if $x_n \to x \in \partial P$, then $\| Du_a(x_n) \| \to \infty$. By the concavity of u_a this implies that for any $\epsilon > 0$ there is $\delta > 0$ such that if $y \in \{x \in P: x^j \leq \delta$ for some $j\}$, then $\partial_j u_a(y) > 1/\epsilon$ and $y^j < \epsilon$ for some j.

Take any $a \in \mathscr{A}^*$ and $q \in R_+^l$; we claim there is $z_{q,a} \in P$, $z_{q,a} \leq se$ such that whenever $x_n \to x$, $x_n \in P$, $x_n \leq se$, we have $u_a(z_{q,a}) - qz_{q,a} > \lim u_a(x_n) - qx$. Indeed, it suffices for this, given the strict concavity of u_a, that $u_a(z) - qz$ have a maximizer on $\{x \in P: x \leq se\}$, which is clearly true from the observation of the first paragraph.

Suppose the lemma is false. Then $a_n \to a \in \mathscr{A}^*$, $x_n \to x \in \partial P$, $x_n \leq se$, $\| Du_n(x_n) \| \leq s'$. We can assume $q_n = Du_n(x_n) \to q \neq 0$. For every $z \in \{y \in P: y \leq se\}$ we have $u_n(x_n) + q_n(z - x_n) \geq u_n(z)$, which yields $\lim u_n(x_n) - qx \geq u(z) - qz$. Since u_n converges to u (C^0) uniformly, $\lim u(x_n) - qx \geq u(z) - qz$. Since this should hold for $z = z_{q,a}$, we get a contradiction. ∎

LEMMA 2. *Let $x_n : I_n \to P$ be a sequence such that* $\lim_n \inf_i \| x_{ni} \| \to \infty$. *Then* $\lim_n (1/n) \| \sum_{i \in I_n} x_{ni} \| \to \infty$.

Proof. This is standard. If the conclusion is false, then we can assume $\sum_{i \in I_n} (x_{ni}/n) \to z$. For every $1 \leq j \leq l$ and n, let $I_n^j = \{i \in I_n : x_{ni}^j < 2lz^j\}$. Then, for n large enough, $\#(I_n^j)/n > (l - 1)/l$. Hence, for every n large enough, there is $i \in I$ with $\| x_{ni} \| \leq 2l \| z \|$, which is a contradiction. ∎

LEMMA 3. *Given $k > 0$ there is $\mu > 0$ such that for all $\mathscr{E}$, λ, $C \subset I$, and $z \in P$ with $z \geq \#(C)ke$ we have $\| q(\mathscr{E}, \lambda, C, z) \| \leq | \lambda | \mu$.*

Proof. Pick $\mu > Ml^{1/2}/k$. Suppose that for a $q \in P$, $q^j > | \lambda | (\mu/l^{1/2})$. Then $x_i^j(q) < k$ for all i (otherwise, $u_i(x_i(q)) > M$ for some i because $\lambda_i Du_i(x_i(q)) = q$ and u_i is concave) and so $\sum_{i \in C} x_i^j(q) < z^j$. Hence $q \neq q(\mathscr{E}, \lambda, C)$ and we can conclude $\| q(\mathscr{E}, \lambda, C) \| \leq | \lambda | \mu$. ∎

LEMMA 4. *For any $r > 0$ there is a compact set $J \subset S$ such that, for all $\mathscr{E}$, λ, and $C \subset I$ satisfying* $\min_i \lambda_i \geq r | \lambda |$ *we have* $q(\mathscr{E}, \lambda, C)/\| q(\mathscr{E}, \lambda, C) \| \in J$.

Proof. Let $q(\mathscr{E}_n, \lambda_n, C_n)/\| q(\mathscr{E}_n, \lambda_n, C_n)\| \rightarrow p \in S$. By Lemma 2 there is i_n such that, letting $q_n = q(\mathscr{E}_n, \lambda_n, C_n)$, $\{x_{i_n}(q_n)\}_{n=1}^{\infty}$ is a bounded set. So, we can assume $x_{i_n}(q_n) \rightarrow x \in \bar{P}$. Since $q_n = \lambda_{ni_n} Du_{i_n}(x_{i_n}(q_n))$ and $\lambda_{ni_n} \geq r|\lambda_n|$ we have, by Lemma 3, $\| Du_{i_n}(x_{i_n})\| \leq \mu/r$ for some μ. Therefore, $x \in P$ by Lemma 1. Again by the compactness of $\mathscr{A}^*$ we can assume $a_{i_n} \rightarrow a \in \mathscr{A}^*$, and then, by continuity, $p = Du_a(x)/\| Du_a(x)\|$. Since $Du_a(x) \gg 0$ we conclude $p \in S$. ▮

Lemma 5. *Let $Q \subset P$ be a compact region. There is $\epsilon > 0$ and $K > 0$ such that for all $\mathscr{E}$, λ, C, and $z_1, z_2 \in P$ with $(1/\#C)\| \sum_{i\in C} \omega_i - z_h \| \leq \epsilon$ and $x_i(q(\mathscr{E}, \lambda, C, z_h)) \in Q$ for every $i \in C$ and $h = 1, 2$, we have*

$$
\begin{aligned}
| V(\mathscr{E}, \lambda, C, z_1) - V(\mathscr{E}, \lambda, C, z_2) - q(\mathscr{E}, \lambda, C, z_2)(z_1 - z_2)| \\
\leq | \lambda | K \| z_1 - z_2 \|^2/\#C.
\end{aligned}
$$

Proof. Denote $V(\mathscr{E}, \lambda, C, z_h) = V(z_h)$, $q(\mathscr{E}, \lambda, C, z_2) = q$, $\#C = m$. We have $q = DV(z_2)$ and since V is concave, $V(z_1) - V(z_2) - q(z_1 - z_2) \leq 0$.

Let $Q \subset U$, where U is open and convex and $\bar{U} \subset P$ is compact. Pick $\epsilon > 0$ so that, for all $i \in C$, $x_i(q) + ((z_1 - z_2)/m) \in \bar{U}$. Let $K = \frac{1}{2} \max_{x\in \bar{U}, a\in\mathscr{A}^*} \| D^2u_a(x)\|$.

By definition of V, $V(z_1) \geq \sum_{i\in C} \lambda_i u_i(x_i(q) + (z_1 - z_2)/m)$. By Taylor's formula (see, for example, [1, p. 124]),

$$
\begin{aligned}
&\lambda_i u_i \left(x_i(q) + \frac{z_1 - z_2}{m}\right) \\
&\quad = \lambda_i u_i(x_i(q)) + q\frac{z_1 - z_2}{m} + \frac{\lambda_i}{2m^2}(z_1 - z_2)\, D^2u_i(y_i)(z_1 - z_2),
\end{aligned}
$$

where $y_i \in \bar{U}$. Henceforth,

$$
0 \geq V(z_1) - V(z_2) - q(z_1 - z_2) \geq \frac{1}{2m^2} \sum_{i\in C} \lambda_i(z_1 - z_2)\, D^2u_i(y_i)(z_1 - z_2),
$$

and so,

$$
\begin{aligned}
| V(z_1) - V(z_2) - q(z_1 - z_2)| &\leq \frac{| \lambda |}{2m^2} m2K \| z_1 - z_2 \|^2 \\
&= \frac{| \lambda |}{m} K \| z_1 - z_2 \|^2. \quad ▮
\end{aligned}
$$

Lemma 6. *Let $Q \subset P$ be a compact region. Then there is $\epsilon > 0$ and $K > 0$ such that for all $\mathscr{E}$, λ, C, $z \in P$ with $(1/\#C)\| \sum_{i\in C} \omega_i - z \| \leq \epsilon$ and $x_i(q(\mathscr{E}, \lambda, C)) \in Q$ for every i, we have $\| D_z^2 V(\mathscr{E}, \lambda, C, z)\| \leq 2 | \lambda | K/\#C$.*

Proof. Let ϵ and K be as in Lemma 4. Denote $D_z^2V(\mathscr{E}, \lambda, C, z) = A$, $\#C = m$. By Lemma 5 and Taylor's formula we have

$$|v'Av| \leq (2|\lambda|K/m)\|v\|^2 \qquad \text{for all} \quad v \in R^l.$$

But this implies $\|A\| \leq 2|\lambda|K/m$ (because, being A symmetric, $\|A\| \leq |\mu_j|$, where μ_j is the eigenvalue of A with largest absolute value; obviously, $|\mu_j| \leq 2|\lambda|K/m$). ∎

LEMMA 7. *There is $\xi > 0$ such that, for all $\mathscr{E}$, λ, $C \subset I$ and $i \in I$,*

$$|V(\mathscr{E}, \lambda, C \cup \{i\}) - V(\mathscr{E}, \lambda, C)| \leq |\lambda|\xi.$$

Proof. Let $k > 0$ be such that $\omega_a \geq ke$ for all $a \in \mathscr{A}^*$. Obviously, we can restrict ourselves to C's with $\#C \geq m$, where m is chosen so that $\omega_i/m < (k/2)e$. For notational economy we suppress reference to $\mathscr{E}$, λ. By Lemma 2 there is $\mu > 0$ such that, for C with $\#C \geq m$ and $i \in I$, $\|q(C \cup \{i\}, \sum_{j \in C} \omega_j)\| \leq |\lambda|\mu$.

Pick any $C \subset I$ with $\#C \geq m$ and $i \in I$ with $i \notin C$. Denote $\omega = \sum_{j \in C} \omega_j$ and $q = q(C \cup \{i\}, \omega)$. We have

$$|V(C \cup \{i\}) - V(C)| \leq |V(C \cup \{i\}) - V(C \cup \{i\}, \omega)| + |V(C \cup \{i\}, \omega) - V(C)|.$$

By the concavity of V with respect to z, $0 \leq V(C \cup \{i\}) - V(C \cup \{i\}, \omega) \leq q\omega_i$. Since $\sum_{j \in C} x_j(q) \leq \omega$, we have $V(C) \geq \sum_{j \in C} \lambda_j u_j(x_j(q))$ and therefore,

$$0 \leq V(C \cup \{i\}, \omega) - V(C) \leq V(C \cup \{i\}, \omega) - \sum_{j \in C} \lambda_j u_j(x_j(q)) = \lambda_i u_i(x_i(q)).$$

Summing up,

$$|V(C \cup \{i\}) - V(C)| \leq |q\omega_i| + \lambda_i |u_i(x_i(q))| \leq |\lambda|\mu\|\omega_i\| + |\lambda|M. \quad ∎$$

LEMMA 8. *Let $Q \subset P$ be a compact region. Then there is $K > 0$ and $\epsilon > 0$ such that for $\mathscr{E}$, λ, C satisfying $x_i(q(\mathscr{E}, \lambda)) \in Q$ for all $i \in I$ and*

$$\Big\|\sum_{i \in C} (x_i(q(\mathscr{E}, \lambda)) - \omega_i)\Big\| \Big/ \#C < \epsilon,$$

we have

$$\|q(\mathscr{E}, \lambda) - q(\mathscr{E}, \lambda, C)\| \leq |\lambda| K \left(\Big\|\sum_{i \in C} (x_i(q(\mathscr{E}, \lambda)) - \omega_i)\Big\| \Big/ \#C\right).$$

Proof. This is a trivial consequence of Lemma 5. Let $\epsilon > 0$ and $K > 0$ be given by Lemma 6 and put $V(y) = V(\mathscr{E}, \lambda, C, y)$, $x = \sum_{i \in C} x_i(q(\mathscr{E}, \lambda))$,

$\omega = \sum_{i\in C} \omega_i$. Then $q(\mathscr{E}, \lambda) = DV(x)$ and $q(\mathscr{E}, \lambda, C) = DV(\omega)$. So,

$$\| q(\mathscr{E}, \lambda) - q(\mathscr{E}, \lambda, C)\| \leqslant \max_{\substack{z = \alpha x+(1-\alpha)\omega \\ 0\leqslant\alpha\leqslant 1}} \| D^2V(\omega)\| \, \| x - \omega \| \leqslant | \lambda | K \frac{\| x - \omega \|}{\#C}. \quad \blacksquare$$

LEMMA 9. *Let $Q \subset P$ be a compact region. Then there is $K > 0$ and $\epsilon > 0$ such that for all $\mathscr{E}$, λ, C and $i \in I$ satisfying $x_j(q(\mathscr{E}, \lambda)) \in Q$ for all j and (denoting $b_1 = (1/\#C) \| \sum_{j\in C} (x_j(q(\mathscr{E}, \lambda)) - \omega_j)\|$, $b_2 = (1/\#(C \cup \{i\})) \| \sum_{j\in C\cup\{i\}} (x_j(q(\mathscr{E}, \lambda) - \omega_j)\|$, $b = b_1 + b_2$) b_1, $b_2 \leqslant \epsilon$, we have (denoting $q = q(\mathscr{E}, \lambda)$, $q(C) = q(\mathscr{E}, \lambda, C)$, etc.):*

$$\begin{aligned}\eta = | \lambda_i u_i(x_i(q(C \cup \{i\}))) - \lambda_i u_i(x_i(q)) \\ -q(\omega_i - x_i(q)) + q(C)(\omega_i - x_i(q(C \cup \{i\})))| \leqslant | \lambda | Kb.\end{aligned}$$

Proof. For some $r > 0$ if we have $x_j(q(\lambda)) \in Q$ for all j, then $\min_j \lambda_j \geqslant r|\lambda|$. Since η is linearly homogeneous in λ we can assume $| \lambda | = 1$. We have then

$$\begin{aligned}\eta \leqslant\ & | u_i(x_i(q(C \cup \{i\}))) - u_i(x_i(q))| \\ &+ | q(\omega_i - x_i(q)) - q(C)(\omega_i - x_i(q(C)))| \\ &+ \| q(C)\| \, \| x_i(q(C \cup \{i\})) - x_i(q(C))\| \\ \leqslant\ & H_1 \| q(C \cup \{i\}) - q \| + H_2 \| q - q(C)\| + H_3 \| q(C \cup \{i\}) - q(C)\|,\end{aligned}$$

where (use compactness and $\min_j \lambda_j \geqslant r$) H_1, H_2, and H_3 can be chosen independently of $\mathscr{E}$, λ, and C (as long as $| \lambda | = 1$). Since, by Lemma 8, $\| q - q(C \cup \{i\}) \| \leqslant Kb_2$ and $\| q - q(C)\| \leqslant Kb_1$, we get $\eta \leqslant K(H_1 + H_2 + H_3)(b_1 + b_2)$. $\blacksquare$

LEMMA 10. *$\Phi_{\mathscr{E}_n}$ converges to Φ_ν in the sense of C^1 uniform convergence on compacta; i.e., if $J \subset S$ is compact, then*

$$\max_{p\in J} \{\| \Phi_{\mathscr{E}_n}(p) - \Phi_\nu(p)\| + \| D\Phi_{\mathscr{E}_n}(p) - D\Phi_\nu(p)\|\} \to 0.$$

Proof. This is an immediate corollary of Dierker's results [13, Lemma 3, p. 53].

LEMMA 11. *Let Q be a compact region. Then there is $K > 0$ and $\epsilon > 0$ such that for all n, λ, and λ' satisfying $x_i(q(\mathscr{E}_n, \lambda))$, $x_i(q(\mathscr{E}_n, \lambda')) \in Q$ for all $i \in I_n$, we have*

$$\| q(\mathscr{E}_n, \lambda) - q(\mathscr{E}_n, \lambda')\| \leqslant K | \lambda - \lambda' |.$$

Proof. Note first that, by the usual compactness argument, there is $r > 0$, such that for any n, if $x_i(q(\mathscr{E}_n, \lambda)) \in Q$ for all i, then $\lambda_i \geqslant r \,|\lambda|$ for all i. Let $\mu > 0$ be as in Lemma 3 and pick δ, H such that

$$0 < \delta < \{v[D^2u_a(x)]^{-1}v : a \in \mathscr{A}^*, x \in Q, \|v\| = 1\},$$

$$H > (1/|\lambda|)\{\|[D^2u_a(x)]^{-1}q\| : a \in \mathscr{A}^*, x \in Q, \|q\| \leqslant |\lambda|\,\mu\}.$$

From now on we consider a fixed n and suppress reference to $\mathscr{E}_n$. It is also convenient to write $x_i(\lambda_i, q)$ instead of $x_{a_i,\lambda_i}(q)$. We will show that $q(\lambda)$ is C^1 and, for λ satisfying the condition of the lemma, $\|Dq(\lambda)\| \leqslant H/\delta$. By the Mean Value Theorem this implies the desired conclusion with $K = H/\delta$.

Define $\Psi(q, \lambda) = \sum_{i \in I_n} x_i(\lambda_i, q)$. Since $q = q(\lambda)$ if and only if $\Psi(q, \lambda) = \sum_{i \in I_n} \omega_i$ and $D_q\Psi(q, \lambda) = \sum_{i \in I_n} (1/\lambda_i)[D^2u_i(x_i(\lambda_i, q))]^{-1}$ is nonsingular, $q(\lambda)$ is C^1, and $Dq(\lambda) = -[D_q\Psi(q(\lambda), \lambda)]^{-1} D_\lambda\Psi(q(\lambda), \lambda)$. Because $D_{\lambda_i}\Psi(q(\lambda), \lambda) = D_{\lambda_i}x_i(\lambda_i, q(\lambda)) = -(1/\lambda_i^2)[D^2u_i(x_i(\lambda_i, q(\lambda))]^{-1} q(\lambda)$, we have $\|D_\lambda\Psi\| \leqslant n(1/(r\,|\lambda|^2))\,H\,|\lambda| = nH/(r\,|\lambda|)$. We also have $vD_q\Psi(q, \lambda)v \geqslant (n\delta/|\lambda|)\|v\|$ for all v so that $|\mu| \geqslant n\delta/|\lambda|$ for any eigenvalue μ of $D_q\Psi(q, \lambda)$, and therefore $\|D_q\Psi(q, \lambda)\| \geqslant n\delta/|\lambda|$. Hence $\|[D_q\Psi(q(\lambda), \lambda)]^{-1}\| \leqslant |\lambda|/(n\delta)$.

We conclude

$$\|D_q(\lambda)\| \leqslant \|[D_q\Psi(q(\lambda), \lambda)]^{-1}\|\,\|D_\lambda\Psi(q(\lambda), \lambda)\| \leqslant H/\delta. \quad \blacksquare$$

2.3. *Boundedness of r-Optimal Allocations*

PROPOSITION 1. *Let $r > 0$ be given. Then there is a compact region $Q \subset P$ such that for all $\mathscr{E}$, $C \subset I$ and λ with* $\min_i \lambda_i \geqslant r\,|\lambda|$, *then $x_i(q(\mathscr{E}, \lambda, C)) \in Q$ for all i.*

Proof. Without loss of generality we can confine ourselves to $C = I$. Let $J \subset S$ be the compact set given by Lemma 4. Suppose we have $\mathscr{E}_n$ and admissible λ_n with, denoting $x_n = x(q(\mathscr{E}_n, \lambda_n))$, $\lim_n \sup_i \|x_{ni}\| = \infty$. Suppose $\|x_{n1}\| \to \infty$. Without loss of generality we can also assume $\|q(\mathscr{E}_n, \lambda_n)\| = 1$. Since the utility functions are concave and uniformly bounded, $q(\mathscr{E}_n, \lambda_n) \in J$ and $\|x_{n1}\| \to \infty$ is possible only if $\|Du_{n1}(x_{n1})\| \to 0$, and $\lambda_{n1} \to \infty$. But then $\lim_n \inf_i \lambda_{ni} = \infty$, which yields $\lim_n \sup_i \|Du_{ni}(x_{ni})\| = 0$ and this $\lim_n \inf_i \|x_{ni}\| = \infty$. By Lemma 2 this contradicts the feasibility of x_n, and we conclude that $\{\sup_i \|x_{ni}\|\}_{n=1}^{\infty}$ is a bounded set. Henceforth, there is $s > 0$ such that $x_i(q(\mathscr{E}, \lambda)) \leqslant se$ for all $\mathscr{E}$, admissible λ and i.

Observe now that, letting μ be as in Lemma 3 and because of Lemma 1,

$$Q = \{x \in P\colon x \leqslant se \text{ and } \|Du_a(x)\| \leqslant \mu/r \text{ for some } a \in \mathscr{A}^*\}$$

is compact. But for any $\mathscr{E}$ if λ is admissible, then, by Lemma 3

$$r \mid \lambda \mid \| Du_i(x_i(q(\mathscr{E}, \lambda)))\| \leqslant \| q(\mathscr{E}, \lambda)\| \leqslant \mid \lambda \mid \mu \qquad \text{for all} \quad i,$$

i.e., $x_i(q(\mathscr{E}, \lambda)) \in Q$. ∎

2.4. *Some Probability Estimates*

Let $r > 0$ be fixed. Its specific value will be determined later on. Let $Q \subset P$ be the compact region given by Proposition 1. Without loss of generality we can assume that Q is of the form $Q = \{x \in P\colon (1/s)\, e \leqslant x \leqslant se\}$.

Given $\epsilon > 0$, n, $x \in Q^{I_n}$, and $i \in I_n$, let $\beta_n(x, i, \epsilon)$ be the probability assigned by i to the set of coalitions C with $\| \sum_{j \in C} x_j - \omega_j \| \geqslant \#(C)\epsilon$ and, of course, $i \notin C$. Then we have

PROPOSITION 2. *For all* $\epsilon > 0$, n, $x \in Q^{I_n}$, $\sum_{i \in I_n} x_i = \sum_{i \in I_n} \omega_i$,

$$\beta_n(x, i, \epsilon) \leqslant (2\ell/n) \left(1\Big/\left(1 - \exp \left\{- \frac{\ell}{2}\left(\frac{\epsilon}{s} - \frac{1}{n-1}\right)^2\right\}\right)\right).$$

Proof. Let $\epsilon > 0$, $x \in Q^{I_n}$, $i \in I_n$ be fixed. Take $1 \leqslant m \leqslant n - 1$. We consider first the family of coalitions C with $i \notin C$ and $\#(C) = m$. Denote $z_j = x_j - \omega_j$, $z(C) = \sum_{j \in C} z_j$. We have

$$\text{Prob}\{\| z(C)/m \| \geqslant \epsilon\} \leqslant \sum_{h=1}^{\ell} \text{Prob}\{| z^h(C)/m | \geqslant \ell^{1/2}\epsilon\}.$$

But

$$\text{Prob} \left\{\left| \frac{z^h(C)}{m} \right| \geqslant \ell^{1/2}\epsilon\right\} \leqslant \text{Prob} \left\{\left| \frac{z^h(C)}{m} + \frac{z_i^h}{n-1} \right| \geqslant \ell^{1/2}\epsilon - \frac{\ell^{1/2}s}{n-1}\right\},$$

and since $-z_i^h/(n-1)$ is the mean of the random variable which gives weight $1/(n-1)$ to $z_j^h \in [-s, s]$, $j \in I_n \backslash \{i\}$, we have

$$\text{Prob}\{\| x^h(C)/m \| \geqslant \epsilon\} \leqslant (2\ell/n)\, e^{-(\ell/2)((\epsilon/s)-(1/(n-1)))^2 m}$$

(see Hoeffding [18, Theorem 1 and Section 6]). Further references for exponential inequalities of this type are Bennet [6] and Bahadur [5].

Therefore,

$$\beta_n(x, i, \epsilon) = \sum_{m=0}^{n-1} \frac{2\ell}{n}\, e^{-(\ell/2)((\epsilon/s)-(1/(n-1)))^2 m} \leqslant \frac{2\ell}{n} \sum_{m=0}^{\infty} e^{-(\ell/2)((\epsilon/s)-(1/(n-1)))^2 m}$$

$$= (2\ell/n) \left(1\Big/\left(1 - \exp \left\{- \frac{\ell}{2}\left(\frac{\epsilon}{s} - \frac{1}{n-1}\right)^2\right\}\right)\right). \quad ∎$$

Observe that the expression

$$1/(1-\exp\{-\tfrac{1}{2}(\tfrac{\epsilon}{s}-\tfrac{1}{n-1})^2\})$$

goes to ∞ as fast as

$$\left(\frac{\epsilon}{s}-\frac{1}{n-1}\right)^2$$

goes to 0. Hence,

COROLLARY. *There is $H > 0$, N, and $\bar{\epsilon} > 0$ such that for all $0 < \epsilon < \bar{\epsilon}$, $n > N$, $x \in \mathscr{Y}_r(\mathscr{E}_n)$, and $i \in I_n$,*

$$\beta_n(x, i, \epsilon) \le H/(n\epsilon^2).$$

2.5. *A Basic Convergence Property*

Let $0 < r < 1$, for the moment still undetermined, be as in Sections 2.3 and 2.4. For every n, put $\Lambda_n = \{\lambda \in R^{I_n} : \lambda_i \ge r\,|\lambda|$ for all $i \in I_n$, and $\sum_{i\in I_n} \lambda_i = n\}$. We have then $1 \le |\lambda| \le 1/r$ for $\lambda \in \Lambda_n$.

Given n we define two functions $F_n, G_n : \Lambda_n \to R^{I_n}$ as follows (denote $q_n(\lambda) = q(\mathscr{E}_n, \lambda)$):

$$F_n(\lambda)_i = q_n(\lambda)(\omega_i - x_i(q_n(\lambda))),$$

$$G_n(\lambda)_i = v_i(\mathscr{E}_n, \lambda) - \lambda_i u_i(x_i(q_n(\lambda)))).$$

The functions F_n, G_n measure, respectively, the departures of optimal "incomes" from the incomes obtained via initial endowments and of value utilities from optimal feasible ones.

PROPOSITION 3. $\mathrm{Max}_{\lambda\in\Lambda_n} |F_n(\lambda) - G_n(\lambda)| = O(1/n^{1/3})$.

Proof. Let n and $\lambda \in \Lambda_n$ be fixed. In the following we will repeatedly appeal to Lemmata 1–11 and Proposition 2. It is understood that this is legitimate for all, but possibly finitely many, n's. We suppress the index n from now on.

Take any $i \in I$; it will remain fixed. Let $\mathscr{C}$ be the set of coalitions from $I\backslash\{i\}$. Partition $\mathscr{C}$ in three subsets:

$\mathscr{C}_1 = \{C \in \mathscr{C} : \#(C) \le n^{1/3}\}$;

$\mathscr{C}_2 = \left\{C \in \mathscr{C} : \#(C) > n^{1/3} \text{ and } (1/\#C)\left\|\sum_{j\in C}(x_j(q(\lambda)) - \omega_j)\right\| \le 1/n^{1/3}\right\}$;

$\mathscr{C}_3 = \mathscr{C}\backslash\mathscr{C}_1 \cup \mathscr{C}_2$.

Let $C \in \mathscr{C}$ be fixed. We first estimate (suppressing reference to $\mathscr{E}$): $\Psi(C) = V(\lambda, C \cup \{i\}) - V(\lambda, C) - \lambda_i u_i(x_i(q(\lambda))) - q(\lambda)(\omega_i - x_i(q(\lambda)))$. Let $\mu > 0$, $\xi > 0$, $K > 0$ be as in Lemmata 3 to 9. By Lemmas 3 and 7 we have $|\Psi(C)| \leq |\lambda|\xi + |\lambda| M + |\lambda| \mu ls = |\lambda| T \leq T/r$. If $C \in \mathscr{C}_2$, then, by Proposition 1 and Lemma 5, we have (suppressing the index λ):

$$|V(C \cup \{i\}) - V(C) - \lambda_i u_i(x_i(q(C \cup \{i\}))) - q(C)(\omega_i - x_i(q(C \cup \{i\})))|$$

$$= \Big| V\Big(C, \sum_{j \in C} \omega_j + \omega_i - x_i(q(C \cup \{i\}))\Big) - V(C)$$

$$- q(C)(\omega_i - x_i(q(C \cup \{i\}))) \Big|$$

$$\leq |\lambda| K \frac{\| \omega_i - x_i(q(C \cup \{i\}))\|}{\# C} \leq \frac{K}{r} \frac{ls}{n^{1/3}},$$

so that, using Lemma 9,

$$|\Psi(C)| \leq \frac{K}{r}\left(\frac{2}{n^{1/3}} + \frac{ls}{n^{1/3}}\right) + \frac{K}{r}\frac{ls}{n^{1/3}} = \frac{2K}{r}\left(\frac{1 + ls}{n^{1/3}}\right).$$

Let $\pi(C)$ be the probability given by i to coalition C. Then $G_n(\lambda)_i - F_n(\lambda)_i = \sum_{C \in \mathscr{C}} \pi(C) \Psi(C)$.

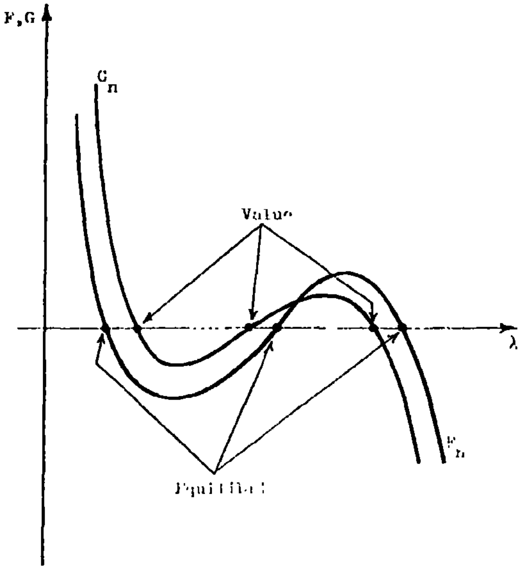

FIGURE 1

Therefore, letting H, γ be as in the corollary to Proposition 2 (and putting $\epsilon = 1/n^{1/3}$):

$$G_n(\lambda) \le F_n(\lambda) \le \sum_{C\in\mathscr{C}} \pi(C)\,\Psi(C) \le \frac{T}{r}\Big(\sum_{C\in\mathscr{C}_1} \pi(C) + \sum_{C\in\mathscr{C}_3} \pi(C)\Big) + 2\,\frac{K}{r}\Big(\frac{1+\sqrt{s}}{n^{1/3}}\Big)$$

$$\le \frac{T}{r}\Big(\frac{1}{n^{1/3}} + \frac{1}{n^{1/3}+\gamma}\Big) + 2\,\frac{K}{r}\Big(\frac{1+\sqrt{s}}{n^{1/3}}\Big) = O\Big(\frac{1}{n^{1/3}}\Big). \quad \blacksquare$$

Figure 1 gives an intuitive account of the proof of the theorem to be carried out in Sections 2.6 and 2.7 by using Proposition 3 and the regularity of the limit economy.

2.6. *Value Allocations Are Close to Competitive Allocations*

Let r be as in Section 5.

PROPOSITION 4. $\rho(\mathscr{V}_r(\mathscr{E}_n), \mathscr{W}(\mathscr{E}_n)) = O(1/n^{1/3})$.

Proof. Let $x_n \in \mathscr{V}_r(\mathscr{E}_n)$, $x_n = x_n(q(\mathscr{E}_n, \lambda_n))$, $q(\mathscr{E}_n, \lambda_n) = q_n$, $\|q_n\| = 1$. Without loss of generality we can assume that $q_n \to p \in J$, where J is as in Lemma 4. We will show the existence of $y_n \in \mathscr{W}(\mathscr{E}_n)$ such that $\max_{i\in I_n} \|x_{ni} - y_{ni}\| = O(1/n^{1/3})$; it is a trivial matter to verify that every bounding estimate can be chosen uniformly, i.e., independently of the particular x_n considered; so $O(1/n^{1/3})$ will not, in fact, depend on x_n, which yields the proposition.

By Proposition 3, $F_n(\lambda_n) = O(1/n^{1/3})$, i.e., $\max_{i\in I_n} |q_n(\omega_i - x_i(q_n))| = O(1/n^{1/3})$. So, $\max_{i\in I_n} \|\varphi(a_i, q_n, q_n\omega_i) - \varphi(a_i, q_n, q_n x_i(q_n))\| = O(1/n^{1/3})$ and, therefore, $(1/n)\|\sum_{i\in I_n} (\varphi(a_i, q_n, q_n\omega_i) - \omega_i)\| = \|\Phi_n(q_n)\| = O(1/n^{1/3})$.

Since $\mathscr{E}_n \to \nu$ and $q_n \to p$ we have $p \in \Pi(\nu)$ (see, for example, [13, Proposition 3, p. 49]). Since ν is regular, there are (except possibly for finitely many n's) $p_n \in S$ such that $p_n \in \Pi(\mathscr{E}_n)$ (of course, $p_n \in J$).

Suppose that $\|\Phi_n(q_n)\|/\|p_n - q_n\| \to 0$. By the Mean Value Theorem $\|\Phi_n(q_n)\| = \|D\Phi_n(q_n')(q_n - p_n)\|$, where $q_n' \in S$ belongs to the (strictly speaking, curvilinear) segment $[q_n, p_n]$. Therefore,

$$\|D\Phi_n(q_n')(q_n - p_n)/\|q_n - p_n\|\| \to 0.$$

We can assume that $(q_n - p_n)/\|q_n - p_n\| \to v$; of course, $v \neq 0$ and $v \in T_p(S)$. Since Φ_n converges C^1 continuously to Φ_ν (Lemma 10), we have $D\Phi_\nu(p)v = 0$, but this is impossible because the regularity of ν means precisely that $D\Phi_\nu(p)$ maps $T_p(S)$ onto itself. This contradiction proves that, for some $H > 0$, $\|p_n - q_n\| \le H\|\Phi_n(q_n)\|$ and therefore $\|p_n - q_n\| = O(1/n^{1/3})$ which yields, by the compactness of $\mathscr{A}^*$ and J, $\max_{i\in I_n} \| \varphi(a_i, p_n, p_n\omega_i) - \varphi(a_i, q_n, q_n\omega_i)\| = O(1/n^{1/3})$.

The proof is finished, since, remembering $x_i(q_n) = p(a_i, q_n, q_n x_i(q_n))$,

$$\max_{i\in I_n} \| \varphi(a_i, p_n, p_n\omega_i) - x_i(q_n)\|$$

$$\leq \max_{i\in I_n} \| \varphi(a_i, p_n, p_n\omega_i) - \varphi(a_i, q_n, q_n\omega_i)\|$$

$$+ \max_{i\in I_n} \| \varphi(a_i, q_n, q_n\omega_i) - \varphi(a_i, q_n, q_n x_i(q_n))\| = O(1/n^{1/3});$$

so take $y_{ni} = \varphi(a_i, p_n, p_n\omega_i)$. ▮

2.7. *Competitive Allocations Are Close to Value Allocations*

Let r be as in the preceding sections. The crucial fact here is:

Proposition 5. *There is $c > 0$ such that for all (but possibly finitely many) n and $\lambda, \lambda' \in \Lambda_n$ with $F_n(\lambda) = 0$, $|\lambda - \lambda'| \leq c$: $|F_n(\lambda')| \geq c\,|\lambda' - \lambda|$.*

Proof. Given any $\lambda \in \Lambda_n$ denote $\hat{\lambda} = \lambda/\|q(\lambda)\|$; then $\|q(\hat{\lambda})\| = 1$. Clearly, $\{\|q(\lambda)\| : \lambda \in \Lambda_n\}$ can be enclosed in a compact set. Let $J \subset S$ be as in Lemma 4.

Suppose that the proposition is false, i.e., $n \to \infty$, $F_n(\lambda_n) = 0$, $\lambda_n', \lambda_n \in \Lambda_n$, $\lambda_n \neq \lambda_n'$, $|\lambda_n - \lambda_n'| \to 0$, and $|F_n(\lambda_n')|/|\lambda_n - \lambda_n'| \to 0$. Clearly, $|F_n(\hat{\lambda}_n')|/|\lambda_n - \lambda_n'| \to 0$. We can also assume that $q(\hat{\lambda}_n) \to p \in S$; since $F(\lambda_n) = 0$ implies $\Phi_n(q(\hat{\lambda}_n)) = 0$, we have $p \in \Pi(\nu)$.

We show now that $|\hat{\lambda}_n - \hat{\lambda}_n'|/|\lambda_n - \lambda_n'| \to 0$ is impossible. Suppose it did occur. Denote $d_n = 1/\|q(\lambda_n)\|$, $d_n' = 1/\|q(\lambda_n')\|$. Then $|d_n\lambda_n - d_n'\lambda_n'|/|\lambda_n - \lambda_n'| \to 0$ and $(1/n)\,|\sum_i (d_n\lambda_{ni} - d_n'\lambda'_{ni})| = |d_n - d_n'|$, which implies $|d_n\lambda_n - d_n'\lambda_n'| \geq |d_n - d_n'|$ and this $|d_n - d_n'|/|\lambda_n - \lambda_n'| \to 0$, or $(d_n - d_n')/|\lambda_n - \lambda_n'| \to 0$. However, $d_n\lambda_n - d_n'\lambda_n' = \lambda_n(d_n - d_n') + d_n'(\lambda_n - \lambda_n')$, yielding

$$\frac{|d_n\lambda_n - d_n'\lambda_n'|}{|\lambda_n - \lambda_n'|} = \left| \lambda_n \frac{d_n - d_n'}{|\lambda_n - \lambda_n'|} + d_n' \frac{\lambda_n - \lambda_n'}{|\lambda_n - \lambda_n'|} \right| \to \lim_n d_n' \neq 0.$$

Henceforth, we have $|F_n(\hat{\lambda}_n')|/|\hat{\lambda}_n - \hat{\lambda}_n'| \to 0$.

By Taylor's formula, for some $H_1 > 0$,

$$\|\Phi_n(q(\hat{\lambda}_n'))\| = (1/n) \left| \sum_{i\in I_n} (\varphi(a_i, q(\hat{\lambda}_n'), F_{ni}(\hat{\lambda}_n') + q(\hat{\lambda}_n')(x_i(q(\hat{\lambda}_n'))) - \omega_i) \right|$$

$$\leq (H_1/n) \sum_{i\in I_n} |F_{ni}(\hat{\lambda}_n')| \leq H_1\,|F_n(\hat{\lambda}_n')|.$$

By Lemma 11 $q(\hat{\lambda}_n') \to p$; indeed, $\|q(\lambda_n) - q(\lambda_n')\| \to 0$ (Lemma 11), which implies $\|q(\hat{\lambda}_n) - q(\hat{\lambda}_n')\| = (1/\|q(\lambda_n)\|)\,\|q(\lambda_n) - (\|q(\lambda_n)\|/\|q(\lambda_n')\|)\,q(\lambda_n')\| \to 0$, because, for any $\lambda \in \Lambda_n$,

$$0 < r \min_{\substack{u\in\mathscr{A}^* \\ x\in Q}} \|Du_u(x)\| \leqslant \|q(\lambda)\|.$$

We have, therefore, $q(\hat{\lambda}_n) \to p \in \Pi(\nu)$, $q(\hat{\lambda}_n') \to p \in \Pi(\nu)$, $\Phi_n(q(\hat{\lambda}_n)) = 0$, and ν regular. As in the proof of Proposition 4, this implies $\|q(\hat{\lambda}_n) - q(\hat{\lambda}_n')\| \leqslant H_2 \|\Phi_n(q(\hat{\lambda}_n'))\|$ for some $H_2 > 0$. We conclude $\|q(\hat{\lambda}_n) - q(\hat{\lambda}_n')\|/|\hat{\lambda}_n - \hat{\lambda}_n'| \to 0$.

Let $H_3 = \max_{a\in\mathscr{A}^*,\, p\in J} \|D_p(1/\|Du_a(\varphi(a, p, p\omega_a))\|)\|$. Without loss of generality, we can assume $|\hat{\lambda}_{n1} - \hat{\lambda}'_{n1}| \geqslant |\hat{\lambda}_n - \hat{\lambda}_n'|$. Define

$$\xi_n(\eta, w) = 1/\|Du_{n1}(\varphi(a_{n1}, q(\eta), w))\|.$$

Then $\hat{\lambda}_{n1} = \xi_n(\hat{\lambda}_n, q(\hat{\lambda}_n)\,\omega_{n1})$, $\hat{\lambda}'_{n1} = \xi_n(\hat{\lambda}_n', q(\hat{\lambda}_n')\,\omega_{n1} - F_n(\hat{\lambda}_n')_1)$; define $\lambda''_{n1} = \xi_n(\hat{\lambda}_n', q(\hat{\lambda}_n')\,\omega_{n1})$. Since $|F_{n1}(\hat{\lambda}_n')|/|\hat{\lambda}_{n1} - \hat{\lambda}'_{n1}| \to 0$, an application of Taylor's formula yields $|\hat{\lambda}_{n1} - \hat{\lambda}'_{n1}|/|\hat{\lambda}_{n1} - \lambda''_{n1}| \to 1$. On the other hand, the Mean Value Theorem gives $|\hat{\lambda}_{n1} - \lambda''_{n1}| \leqslant H_3 \|q(\hat{\lambda}_n) - q(\hat{\lambda}_n')\|$, and so we can assume $|\hat{\lambda}_{n1} - \hat{\lambda}'_{n1}| \leqslant H_3 \|q(\hat{\lambda}_n) - q(\hat{\lambda}_n')\|$, but this implies $|\hat{\lambda}_{n1} - \hat{\lambda}'_{n1}|/|\hat{\lambda}_n - \hat{\lambda}_n'| \to 0$, contradicting $|\hat{\lambda}_{n1} - \hat{\lambda}'_{n1}| \geqslant |\hat{\lambda}_n - \hat{\lambda}_n'|$. ∎

Suppose that the $0 < r < 1$ of the previous sections has been chosen small enough to have

$$\min_{\substack{a\in\mathscr{A}^* \\ p\in\Pi(\nu)}} \frac{1}{\|Du_a(\varphi(a, p, p\omega_a))\|} > r \max_{\substack{a\in\mathscr{A}^* \\ p\in\Pi(\nu)}} \frac{1}{\|Du_a(\varphi(a, p, p\omega_a))\|}.$$

Let $H > 0$ be such that, for n large enough, if $\lambda \in \Lambda_n$, then $|F_n(\lambda) - G_n(\lambda)| \leqslant H/n^{1/3}$ (Proposition 3). Denote $\delta_n = 2H/(cn^{1/3})$, where c is as in Proposition 5.

Let $x_n \in \mathscr{W}(\mathscr{E}_n)$ and $p_n \in S$ be the competitive prices associated with x_n. Without loss of generality we can assume $p_n \to p \in \Pi(\nu)$. If, for $i \in I_n$, we let $\hat{\lambda}_{ni} = 1/\|Du(\varphi(a_i, p_n, p_n\omega_{ni}))\|$, then $p_n = q(\mathscr{E}_n, \hat{\lambda}_n)$ and (for n large enough) $\hat{\lambda}_{ni} > r\,|\hat{\lambda}_n|$ for all $i \in I_n$. If we then define $\lambda_n = (n/(\sum_{i\in I_n} \hat{\lambda}_{ni}))\,\hat{\lambda}_n$, we get $\lambda_n \in \Lambda_n$. Of course, $F_n(\lambda_n) = 0$. Moreover, an easy compactness argument yields the existence of $\epsilon > 0$ (independent of the particular sequence x_n) such that $|\lambda_n - \lambda'| < \epsilon$ and $\sum_{i\in I_n} \lambda_i' = n$ implies $\lambda' \in \Lambda_n$ (again, for n large enough). So, the set $\Gamma_n = \{\lambda \in \Lambda_n : |\lambda - \lambda_n| = \delta_n\}$ is (topologically) a sphere.

By Proposition 5, $\operatorname{Min}_{\lambda\in\Gamma_n} |F_n(\lambda)| \geqslant 2H/n^{1/3}$. Letting, for every n and $\lambda \in \Gamma_n$, $i(\lambda, n)$ be such that $|F_n(\lambda)_{i(\lambda,n)}| \geqslant 2H/n^{1/3}$, we have, for every n and $\lambda \in \Gamma_n$:

$$\begin{aligned} 0 \leqslant |F_n(\lambda)_{i(\lambda,n)} - G_n(\lambda)_{i(\lambda,n)}| &\leqslant |F_n(\lambda) - G_n(\lambda)| \\ &\leqslant H/n^{1/3} < 2H/n^{1/3} \leqslant |F_n(\lambda)_{i(\lambda,n)}|. \end{aligned}$$

Therefore, the maps $F_n \mid \Gamma_n$ are homotopic with respect to $R^{l_n}\setminus\{0\}$; just take $\alpha F_n + (1 - \alpha) G_n$ for a homotopy.

The situation now is as follows: the map F_n is C^1 on Λ_n, and since $\sum_i F_n(\lambda)_i = 0$ it maps at every $\lambda \in \Lambda_n$ into the tangent plane of Λ_n; if we denote $\hat{\Gamma}_n = \{\lambda \in \Lambda_n : |\lambda - \lambda_n| \leq \delta_n\}$, then Γ_n is the boundary (rel. Λ_n) of $\hat{\Gamma}_n$ and, by Proposition 5, $F_n(\lambda) = 0$, $\lambda \in \hat{\Gamma}_n$, implies $\lambda = \lambda_n$; also by Proposition 5 the map $DF_n(\lambda_n)$ is nonsingular. Therefore, degree $F_n \mid \Gamma_n \neq 0$ and since $F_n \mid \Gamma_n$, $G_n \mid \Gamma_n$ are homotopic relative to $R^{l_n}\setminus\{0\}$, we conclude that degree $G_n \mid \Gamma_n \neq 0$. This implies the existence of some $\lambda_n' \in \hat{\Gamma}_n$ (i.e., $|\lambda_n - \lambda_n'| < 2H/(cn^{1/3})$) with $G_n(\lambda_n') = 0$ (for the topological results used in this paragraph, see Guillemin and Pollack [14, Chaps. 2, 14]).

The conclusion of the previous paragraph yields $\rho(\{\lambda \in \Lambda_n : F_n(\lambda) = 0\}, \{\lambda \in \Lambda_n : G_n(\lambda) = 0\}) = O(1/n^{1/3})$. Since $r > 0$ was chosen sufficiently small for $\{\lambda \in \Lambda_n : F_n(\lambda) = 0\}$ to contain all competitive equilibria, it is immediate from Lemma 11 and the usual compactness argument that this implies $\rho(\mathscr{W}'(\mathscr{E}_n), \mathscr{V}'_r(\mathscr{E}_n)) = O(1/n^{1/3})$. Q.E.D.

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JOURNAL OF ECONOMIC THEORY **14**, 439–442 (1977)

Notes, Comments, and Letters to the Editor

The St. Petersburg Paradox: A Con Game?*

Careful analysis of the "St. Petersburg" lottery reveals no logical or mathematical absurdity inherent in risk neutrality for money. There is an empirical absurdity, but it rests on an additional, easily overlooked assumption about the gullibility of the gambler that is itself empirically absurd.

Since the time of the Bernoullis,[1] an impression has lingered that the famous "St. Petersburg game" is some kind of counterexample to risk neutrality for money, and that some kind of mathematical or logical contradiction inheres in the use of expected monetary values as a utillity index.[2] If this were, in fact, the case, their use could be not only inaccurate in practice but unsound in principle, and perhaps unsuited even for simplified decision models where small quantitative inaccuracies would ordinarily be tolerated. It is hoped that the following brief discussion of the underlying logic of the "St. Petersburg paradox," reinforced by a few mundane numerical illustrations, will help to lay these doubts to rest.

Let us "walk through" the paradox, step by step:

(I) First, one supposes that a "rational" person has been found whose utility for money happens to be both linear and risk neutral.

(II) One then proposes, in return for his payment of an entrance fee, to toss coins with the subject until he fails to win, and then to pay him 2^k cents, where k (= 0, 1, 2,...) is his total number of wins.

(III) One then calculates, from (I) and (II), that his utility for this proposition must be infinite:

$$\tfrac{1}{2}\cdot 1+\tfrac{1}{4}\cdot 2+\tfrac{1}{8}\cdot 4+\tfrac{1}{16}\cdot 8+\cdots=\infty,$$

and in particular that it exceeds $1,000 (or any other finite amount).

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[1] See especially Daniel Bernoulli's 1738 paper [1]. The paradoxical game (or gambling proposition) was propounded 25 years earlier by Nicolas Bernoulli in a letter to Pierre de Montmort; this letter is reproduced in [1], along with a subsequent analysis of the problem by Gabriel Cramer.

[2] See, for example, the opening sentence of [5].

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The paradox is not yet upon us. An infinite utility may not be absurd, per se, when unboundedly large sums of money are in prospect.[3] Let us move ahead, then, cautiously:

(IV) One now argues, from (III), that the subject will gladly pay \$1,000 (or any other amount) as an entry fee.

(V) One next observes, from experience or introspection, that rational people most certainly do *not* behave as in (IV).

(VI) Finally, one concludes from (I) and (V), that linear risk-neutral utility for money is contrary to experience.

This, then, is the paradox. The reductio ad absurdum is not logical or mathematical, after all, but empirical. Our experience is contradicted. The conclusion is nevertheless impressive. At first glance, it throws serious doubt upon the realism of our starting assumption (I), namely, that rationality and risk linearity for money are compatible human traits.

A closer inspection, however, reveals a weak link in the chain—a fatal flaw that invalidates the conclusion. Despite our show of caution in moving from (I), (II), (III) to (IV), we left implicit one key assumption, without which we cannot claim to enter the mind of the subject and say what he will do, gladly or otherwise. The missing link:

(II-$\frac{1}{2}$) One assumes that the subject believes the offer to be genuine, i.e., believes *that he will actually be paid, no matter how much he may win.*

Something like this is essential. Since the end of the chain is empirical, we must provide (II-$\frac{1}{2}$) to link the fictive experiment to its real context. If it should prove, by the very nature of (II), that no rational person could possibly be convinced that the game is in earnest, then (II-$\frac{1}{2}$) fails, and with it (IV) and the conclusion (VI)[4].

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[3] If the only issue were infinite utility, then we could obtain the desired effect in a probability-free context. Indeed, consider the game of "Blank Check," in which a sponsor offers, in return for a specific entrance fee, to pay the subject any finite amount of money that he names.

[4] See [3, p. 228]. Lord Keynes, in his well-known essay [4], also shows some concern for the believability of the game: "We are unwilling to be Paul, partly because *we do not believe Peter will pay us if we have good fortune in the tossing*, partly because we do not know what we should do with so much money ··· if we won it, partly because we do not believe we ever should win it, and partly because we do not think it would be a rational act to risk an infinite sum or even a very large finite sum for an infinitely larger one, whose attainment is infinitely unlikely" [4, p. 1370] (italics ours). But he apparently does not consider the disbelief in payment to be sufficient in itself to escape the paradox: "... Peter has undertaken engagements which he cannot fulfill; if the appearance of heads is deferred even to the 100th toss, he will owe a mass of silver greater in bulk than the sun. But this is no answer. Peter has promised much and a belief in his solvency will strain our imaginations; but it is imaginable" [4, p. 1368]. Here he seems to be saying that it is difficult, but possible, to

This may seem like nit picking, but it is not. The sting of the infinite series is only in its tail, and our experiment must be able to distinguish the *unbounded* proposal (II) from any bounded replica of it. Imagine for example, that (II-½) falls short—but just barely. Imagine that we have found a suitably rational, suitably risk-linear, and suitably mathematically educated personage, who lacks only the childlike trustfulness of (II-½). He believes, let us say, that the well-dressed stranger with pointed shoes is good for the value of the United States gross national product, say 2^{17} cents, but no more! Then the expected payoff, exclusive of entry fee, works out as follows:

$$\tfrac{1}{2}\cdot 1 + \tfrac{1}{4}\cdot 2 + \tfrac{1}{8}\cdot 4 + \cdots + \frac{1}{2^{47}}\cdot 2^{46} + \frac{1}{2^{47}}\cdot 2^{17} = 47\cdot\tfrac{1}{2} + 1 = 24.5 \text{ cents.}$$

So our paragon of linearity and gullibility would *not* gladly pay $1,000; he would even balk at paying 25 cents for a chance at a year's national output!

Coming down to earth, let us suppose that a gambler is convinced that the casino will pay up to a $10,000 limit on the St. Petersburg game. Then in the penny version (II), the "fair" entrance fee is just about 11 cents. In a dollar game, it would be $7.61. These numbers are *not* absurd on their face. In short, the conclusion (IV) cannot be sustained when confronted by reasonable inputs and beliefs, and the paradox evaporates.[5]

Another way to drive home our point is to focus on the notion of "rationality" that enters into (I) and (V) above. There seems little doubt that anyone who would gladly pay $1,000 to enter the St. Petersburg lottery is a fool. His folly is revealed, however, not in his linear attitude toward monetary gambles for high stakes—an attitude which may be foolish but which is not seriously tested on this occassion—but rather in his blissful confidence that the sponsors of the game can and actually will pay the prize, even to the tune

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imagine a "Paul" for whom the paradox works, in the sense of (II-1/2). We would rather exclude such freakish personalities from our models.

Karl Menger, in his otherwise lucid and well-argued treatment [6], is equally unsatisfactory on this point, sidestepping the credibility question with: "If he is sane, [the gambler will not] risk all or even a considerable portion of his wealth in a St. Petersburg Game" [6, p. 212]. This is similar to Bernoulli's: "any fairly reasonable man would sell his chance with great pleasure, for twenty ducats" [1, p. 31]. Neither author pauses to ask whether a sane man would believe in payoffs without bound.

[5] Menger rather stiffly regards the assumption of a payoff limit as violating the "definition of the game" and as "introducing factors that are outside the problem" [6, p. 214]. This is a curious objection from one who has just taken such care to stress the empirical content of the paradox. Moreover, he seems to regard bounding the payoff as no more than a device to ensure finiteness, saying that is is "unable to explain the remarkable discrepancy between mathematical expectation and actual behavior" (p. 214). Here he quite misses the point, made so well by Gabriel Cramer two centuries before, namely, that imposing even an outlandishly high bound will make the mathematical expectation acceptably small. (See [1, pp. 33–35].)

of $10^{60.201}$ dollars.[6] Compare him with the classic "sucker" who buys the Brooklyn Bridge for $272 (or whatever else he happens to have in his pocket at the time); the latter is also a fool, but not because the Brooklyn Bridge is not worth $272. Is the St. Petersburg paradox then nothing more than a "con" game?

One can, of course, devise other lotteries based on infinite series that diverge even more slowly.[7] They would only strengthen our present thesis, to wit: (1) there is no logical or mathematical absurdity in risk neutrality for money or income, and (2) the empirical "absurdity" that seems to arise depends on an additional, commonly overlooked assumption about the subject's credulity—an assumption that is itself empirically absurd. We conclude that the St. Petersburg game is a flimsy weapon indeed with which to attack the use of expected monetary values in decision theory, game theory, or economics, however vulnerable that usage may be on other counts.[8]

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[6] *This is the "house limit" that yields a $1,000 expected value in the penny game.*

[7] This is done, e.g., by Menger to show that any unbounded utility function—including logarithmic or "Bernoullian" utility—can be confronted by a lottery of the St. Petersburg type having infinite expected utility. (See [6, pp. 217–218]; incidentally, on p. 217, e^{2n} should be e^{2^n}.)

[8] Essentially the same argument was previously made by Fry [2]; we are indebted to R. W. Hamming for this reference.

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The St. Petersburg Paradox: A Discussion of Some Recent Comments*

The recent notes by Brito [1] and by Shapley [3] on the St. Petersburg paradox are provocative and certainly shed new light on this old topic, but they miss the point of the paradox. Essentially, Brito's argument is that the bound on the duration of an individual's life imposes a bound on the amount of commodities that he can consume. The utility of this maximal consumption then provides a bound on the range of the utility function, and with a bounded utility there is no paradox.[1] Shapley, though he agrees that utility may in principle be unbounded, argues that no person or institution is in a position to guarantee payment of arbitrarily large sums, so that in fact there can exist no "reliable" lottery ticket with arbitrarily large payoffs. Effectively, therefore, the utilities that appear in lottery tickets are bounded, and again the paradox disappears.

Neither argument is convincing.[2] Brito assumes that the payoffs to lottery tickets are expressible in terms of commodity bundles of a fixed finite dimension l, and that the utility function is defined and finite on the entire non-negative orthant of l space.[3] Shapley assumes that the lottery ticket represents an obligation to the individual in question—let's call him Paul—undertaken by some other individual or institution, whom we may call Peter. But these assumptions, though they are part of the classical presentation of the paradox, seem irrelevant and artificial. The payoffs need not be expressible in terms of a fixed finite number of commodities, or in terms of commodities at all; and the probabilities in the "lottery ticket" need not represent the promises of Peter, but could simply correspond to Paul's own evaluation of the

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[1] Menger [2, pp. 217 ff.], has pointed out that the paradox is constructible if and only if utility is unbounded. See also [1, p. 123].

[2] Shapley concentrates his fire not so much on the paradox itself, but on the view that it constitutes an argument against risk neutrality for money. In this he is right; though as he himself acknowledges, one does not need the St. Petersburg paradox to reject risk neutrality. But it appears from his paper that he thinks that he has actually resolved the whole paradox; and in this we think he is wrong.

[3] Or alternatively, that the consumption space is compact.

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situation.[4] If this is the case there is nothing to prevent the construction of a lottery ticket with an infinite expected utility, and then the full force of the paradox is upon us.

For example, the lottery ticket that Paul is considering might be some kind of open-ended activity—one that could lead to sensations that he has not heretofore experienced. Examples might be religious, aesthetic, or emotional experiences, like entering a monastery, climbing a mountain, or engaging in research with possibly spectacular results. It seems reasonable to suppose that before engaging in such an activity, Paul would perceive the utility of the resulting sensation as a random variable, and there is no particular reason to assume that this random variable is bounded. Shapley's credibility issue would of course not arise in such a situation.

In my opinion the simplest, most straightforward, and most natural resolution of the paradox lies in the conclusion that utility must be bounded. Unbounded utility would lead to counterintuitive conclusions even without the St. Petersburg paradox. If Paul's utility were unbounded, then for any fixed prospect x (e.g., a long, happy, and useful life), there would be a prospect y with the property that Paul would prefer a lottery yielding y with probability $1/10^{100}$ and death with the complementary probability to the prospect x. This, I think, is about as hard to swallow as the idea of infinite utility.[5]

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[4] It is conceivable that Shapley is merely saying in a roundabout way that resources are bounded; that Paul cannot reasonably assign positive probability to receiving more than the world's GNP, whether he is buying a lottery ticket from Peter or whether he is entering business on his own. In that case, of course, the argument is essentially the same as Brito's.

[5] Basically, this is merely a restatement of the St. Petersburg argument. It is essentially due to Menger [2, p. 221], though he himself appears to have misunderstood its implications. He writes that "most people would refuse to risk one dollar in order to obtain a probability of 1/10,000,000 of winning an amount which has even a subjective value of $10 million." It is of course not clear what is meant by "an amount with a subjective value of $10,000,000." One interpretation might be an amount x whose utility $u(x)$ is 10,000,000 $u(\$1)$. But by the definition of utility, Paul *would* then be willing to risk $1 in order to get a 1/10,000,000 chance at x. One can make sense of the above quotation from Menger only by turning it around to say that for most people there is *no* prospect x for which they would risk $1, when the probability of winning is only 1/10,000,000. But this simply means that utility is bounded by 10,000,000 $u(\$1)$.

Press, Princeton, N. J., 1967. (Translated by Wolfgang Schoellkopf from the *Z. Nationalokonomie* **5** (1934), 349–485.)

3. L. S. SHAPLEY, The St. Petersburg paradox: A con game?, *J. Econ. Theory* **14** (1977), 439–442.

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Lotteries and Menus: A Comment on Unbounded Utilities*

This note, like [16] and [2] before it, busies itself with ramifications of the *St. Petersburg paradox*, that durable and resilient springboard for philosophical speculations about money, risk, utility, and rationality. To some extent we are rebutting [2] and defending [16], but our main purpose here is to expand on some ideas about unbounded utility that were only briefly mentioned in [16] and that are clearly relevant to the issues considered in both [16] and [2].

In retrospect,[1] we can hardly claim that [16] undertakes to "resolve the whole paradox," as [2] suggests. But [16] does undertake to show that Nicholas Bernoulli's famous lottery is not very well suited to carrying out the task most commonly assigned to it—that of making a strictly risk-neutral attitude toward money look absurd.[2] In brief, we argued in [16], first, that the apparent absurdity is empirical, rather than logical or mathematical, and second, that the Bernoulli lottery, taken literally, involves such unbelievable sums of money that one cannot possibly claim an *empirical* basis for concluding that the gambler's decisions based on expected monetary return are counterintuitive or irrational.[3] Finally, by some numerical illustrations, we reminded the reader that if the lottery is truncated, so that only credible sums of money are promised, then decisions based on expected-value calculations are not at all inconsistent with our ordinary intuition and experience as to how "rational" people behave.[4]

Professor Aumann challenges this argument only obliquely in [2], by raising the prior question of whether unbounded utilities—for money or anything else—can ever be entertained in such a context without creating insurmountable difficulties either in logic or in interpretation. This question indeed threatens to foreclose the whole thesis of [16], by denying the very

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[1] Actually, [16] was written several years ago and circulated as a Rand paper; it is now published with only minor revisions.

[2] Thus, Kim opens with the flat statement: "The lesson from the St. Petersburg paradox... has been to reject the assumption of linear utility of income or wealth (and, therefore, the mathematical expectations approach) ..." [10, p. 148].

[3] We were then unaware of Fry's treatment [5, pp. 194–199], which makes substantially the same argument.

[4] This observation goes all the way back to Gabriel Cramer, writing *before* Bernoulli's "St. Petersburg" paper. See also Fellner's discussion [4, p. 106].

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terms of the Bernoulli lottery. In fact, Aumann insists that unboundedness is the "real point" of the St. Petersburg paradox, and that I (and many others) have missed it.

The unboundedness issue was indeed slighted in [16], and this may have led Aumann in [2] to read between the lines and object to what he seemed to think I seemed to think. But there was one brief remark on the subject in [16], which claimed that unbounded utility is *not* the real issue in the Bernoulli lottery and that, in any case, it can be confronted more directly with another, much simpler experiment called "Blank Check." In the present note I shall attempt first to justify the first half of this claim—in a way that may provide common ground on which both [16] and [2] can comfortably stand. Then, in a more constructive spirit, I shall attempt to systematize the point about "Blank Check" and show how the whole anti-unbounded-utility position can be made independent of gambling behavior and, indeed, of cardinal utility itself.

For definiteness, let us recapitulate the Bernoulli lottery, using the "utility" form since money does not concern us at the moment. Let a rational individual be given, equipped with a real-valued utility function u and facing a countable infinity of mutually exclusive prospects $\{x_n : n = 1, 2, \ldots\}$, the utilities of which have no upper bound. By subsequencing and relabeling we may assume without loss of generality that $u(x_{n+1}) \geq 2u(x_n) > 0$ for each n. We can then define a lottery, i.e., a random variable, by assigning to each x_n the probability 2^{-n}. This lottery of course has an infinite expected utility, and so the fun begins.

For clarity, we ought to distinguish two rather different "paradoxes" that can arise. As already noted, the first one (I think I am justified in calling it the *classical paradox*) depends on a sharp contrast between what the expected-value theory of rational behavior predicts and what our real-life experience or introspection predicts.[5] The second paradox, however, is not empirical at all but is embedded in the logic of the theory of cardinal utility under risk. As this theory is typically formulated,[6] a *utility function* is given (or derived) that maps a space of mutually exclusive *prospects* into a space of *utilities*, the latter represented as real numbers. The prospect space is assumed to be closed under the operation of forming lotteries, with compound or multistage lotteries being reduced according to the usual rules for combining probabilities. Moreover, the utility function is assumed (or is derived) to be *risk neutral*, so that the utility of any lottery is equal to the expected value of the utility of its outcome. Other assumptions or conditions may also be imposed on the utility function, but it is not usually required that its range be bounded, a priori. On the other hand, the prospect space is usually not required to be closed under probability mixes over infinite sets. Indeed, such infinite lotteries would at once force the range to be bounded, since otherwise there would be a "Bernoulli" lottery, as above, whose utility is not a real number.[7] Since it is difficult on intuitive or even on practical grounds to rule out infinite lotteries, it seems advisable to rule out unbounded cardinal utilities. Logically we cannot have both.[8]

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[5] We keep returning to Menger's point about *empirical rationality* (see [12]) because it is so easily forgotten when the discussion warms up. Yet, a conscientious presentation of the classical paradox will almost always have this essential ingredient buried somewhere in its logic or language, whether or not the author faces up to the implications. Some examples: Fry: "No *sane man* would ever consider paying the bank one hundred dollars for such a chance" [5, p. 195]; Brito: "... so an *individual should be willing* to pay any finite sum ... A little thought will tell us that *this is not true*; hence the paradox" [3, p. 123]; Arrow: "... he 'should' be willing to pay any finite stake. Introspection tells us that *he will not*, and *this 'fact'* will have to be explained" [1, p. 407]; Nicholson: "Some introspection, however, should convince anyone that *no player would pay very much* ... to play this game. If I charged $1 billion ... *I would surely have no takers*" [13, p. 147]; "... intuitively obvious that no *reasonable person* would prefer ... this game ... to the outright receipt of any appreciable amount of money. *Search your mind, reader, and I am sure you will agree!*" [4, p. 102]. (Italics added.) See also the authors quoted in [16].

One may well ask whether this purely logical argument, resting directly on the "utility" form of Bernoulli's lottery, might not be so devastating that it preempts the classical, empirical paradox. Does it not effectively deny the possibility of the proposed lottery, with its infinitely many outcomes and its unbounded payoffs, and so leave us nothing empirical to talk about? The answer is no, not really. There is a simple way to dodge the problem: finitize the lottery by imposing a "house limit," but take care to set the limit so very, very high that the expected value of the lottery, though finite, is nevertheless outrageously large. Then all the usual classical arguments can be applied.[9] Technical or "mathematical" boundedness is restored by this device, but the force of the empirical paradox, such as it is, remains unabated.

It therefore appears that to dispose of the classical paradox requires not only that utilities be bounded (which is a mathematical statement), but that they somehow be "reasonably bounded" or "not unbelievably large" (which is an empirical statement).[10] If Aumann's final paragraph in [2] is taken in this sense of "reasonable boundedness" (as I believe it must), rather than strict mathematical boundedness, then I quite agree with his remarks as far as they go. My further point in [16], however, is that unreasonably large amounts of *money* must also be disallowed if our everyday experience is to be brought to bear in deciding whether someone's supposedly rational behavior is consistant with a linear utility for wealth or income.[11]

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[6] For example, see [7] or [11].

[7] The extended real line $R \cup \{\infty, -\infty\}$ offers no escape, as the indispensable continuity or "Archimedean" axiom would have to be sacrificed.

[8] Compare the discussions of Hirshleifer [8, p. 228], Isbell [9, p. 360], and Savage [15, p. 95]; the latter, however, seems more concerned with "reasonable boundedness" (see below) than with strict mathematical boundedness.

[9] In [16] we could modify Step (II) to read "... to toss coins with him 10^{100} times or until he fails to win ...," and Step (III) to read "... that his utility for this game is very large, and in particular that it exceeds $1000."

[10] Turning from the *existence* of a bound to its *magnitude* raises the intriguing side question: large compared to what? Both the zero and the unit of cardinal utility are arbitrary, so how can we express the limits of "reasonableness" in an intrinsic or invariant

In order to motivate the rest of this note, consider the following situation. We start with a wealthy sponsor and a rational subject (as in the "money" form of the Bernoulli lottery). The sponsor writes out a check to the subject and signs it, but leaves the amount of money blank. In return for a specified fee or other commitment, he hands over the check and the subject is now free to fill in any amount he pleases and go to the bank to cash the check. (The First Rational Bank of St. Petersburg, perhaps?) We can now ask the obvious, if faintly ridiculous question: What is the blank check worth? What is a reasonable fee, or other consideration?

Serious questions of credibility arise, of course, similar to those discussed in [16]; in this case they are even more starkly exposed to view. But leaving them aside, we can hardly escape the conclusion that if the utility of money is unbounded, then the utility of the blank check is infinite. In other words, this "game," like Bernoulli's, is a way of manufacturing a prospect of infinite utility out of a set of prospects of finite, but unbounded utility. The intriguing feature of the present construction, however, is that we can now argue against unbounded *ordinal* utilities, as no cardinal ideas are involved. Choice has replaced chance.

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way? For one answer, we could take the utility of some fixed prospect (the status quo?) plus a suitably large multiple of the typical or "average" utility difference met with in everyday decisionmaking. Another method might be to follow Aumann in [2] and use as a measuring rod the difference between "a long, happy and useful life" and "death"; we would certainly agree that 10^{100} such units is unreasonably large. Yet another method would be to borrow a leaf from psychometry and postulate not only a "least noticeable difference," but also a "greatest conceivable difference" in utility; the ratio between the two would be independent of scale factors and zero points and might be regarded as an observable personal or sociocultural constant.

[11] In principle, however (as Reinhard Selten has pointed out to us), there should be no reason to exclude empirically unbelievable prospects from at least hypothetical consideration.

"I can't believe *that*," said Alice.

"Can't you?" the Queen said, in a pitying tone. "Try again: draw a long breath, and shut your eyes."

Alice laughed, "There's no use trying," she said: "one *can't* believe impossible things."

"I daresay you haven't had much practice," said the Queen. "When I was your age, I always did it for half-an-hour a day. Why, sometimes I've believed as many as six impossible things before breakfast!"

Let us sketch some highlights of a possible formal approach. Let A denote a domain of mutually exclusive alternatives or prospects, and let $\mathcal{A}$ be a suitably rich family of subsets of A (say, the set of all finite or countable subsets, or any σ-field that includes the singletons). Let "Sel" be a mapping from $\mathcal{A} - \{\phi\}$ to A, with Sel(S) interpreted as a list, or "menu," offering the selection S. More precisely, Sel(S) will represent the *prospect of having to select an outcome from the set S.*[12] The axioms for a "selection space" ($A, \mathcal{A}$, Sel) might then include

$$\operatorname{Sel}(\{x\}) = x, \qquad \text{all } x \in A \tag{A1}$$

(single-choice menus), and

$$\operatorname{Sel}(\{\operatorname{Sel}(S)\colon S \in \mathcal{S}\}) = \operatorname{Sel}\left(\bigcup_{S \in \mathcal{S}} S\right), \qquad \text{all } \mathcal{S} \subset \mathcal{A} \tag{A2}$$

(reduction of compound menus); in the latter we might wish to restrict the sets $\mathcal{S}$ to be countable or even finite collections of subsets of A. Finally, if u is a utility function from A to the reals R, it might be required to satisfy

$$u(\operatorname{Sel}(S)) = \sup_{x \in S} u(x), \qquad \text{all } S \in \mathcal{A}. \tag{A3}$$

This expresses a kind of neutrality: there is neither pain nor pleasure in having to make a choice.

The parallel with cardinal utility theory is striking. To make the analogy, let the domain of prospects now be a measurable space (M, $\mathcal{C}$), and let $\mathbf{M}$ be a suitably rich (and suitably convex) family of probability measures on M, including all the "sure things" δ_x, $x \in M$. Let "Lot" be a mapping from $\mathbf{M}$ to M, and interpret Lot(μ) for $\mu \in \mathbf{M}$ as the *prospect of having to enter the lottery defined by the probability measure* μ. The axioms for a "mixture space" (M, $\mathcal{C}$, $\mathbf{M}$, Lot) might then include

$$\operatorname{Lot}(\delta_x) = x, \qquad \text{all } x \in M \tag{B1}$$

(one-outcome lotteries), and

$$\operatorname{Lot}\left(\int_{\mathbf{M}} \operatorname{Lot}(\mu)\, d\alpha(\mu)\right) = \operatorname{Lot}\left(\int_{\mathbf{M}} \mu\, d\alpha(\mu)\right) \tag{B2}$$

(reduction of compound lotteries); in the latter, α is a probability measure on

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[12] "Sel" should not be confused with a "choice function." Note that generally Sel(S) $\notin S$.

M, perhaps restricted to having a countable or even finite carrier.[13] Finally, of course, we have risk neutrality:

$$u(\mathrm{Lot}(\mu)) = \int_M u(x)\, d\mu(x), \tag{B3}$$

which says there is no pleasure or pain in gambling.[14]

Pursuing the analogy, we recall that a mixture space M is often just the set of all probability measures on a basic set P of "sure" or "pure" outcomes, with the natural definition

$$\mathrm{Lot}(\mu) = \int_{x \in M} x\, d\mu(x), \qquad \text{all } \mu \in \mathbf{M}.$$

If P is finite, then M takes the form of a finite-dimensional simplex.[15] Similarly, a selection space A can be just the set of all nonempty subsets of a basic set Q of no-option or "compulsory" outcomes, with the natural definition

$$\mathrm{Sel}(S) = \bigcup_{x \in Q} x, \qquad \text{all } S \in \mathcal{A}.$$

If Q is finite, then A takes the form of a finite Boolean algebra, omitting the zero.[16]

As already indicated, the Bernoulli game tells us that if we have a mixture space with a sufficiently rich class **M** of admissible "mixtures," then any real-valued utility function satisfying (B3) must be bounded. On the other hand, the "Blank Check" game tells us that if we have a selection space with a sufficiently rich class $\mathcal{A}$ of admissible "menus," then any real-valued utility function satisfying (A3) must be bounded, at least from above. Moreover, the supremum of the range $u(A)$ is actually attained, something that we cannot conclude from the Bernoulli lottery.[17]

To round out the argument, we claim, finally, that *menu forming* is an intrinsically more fundamental process than *lottery forming*, and that it would be far harder to deny (A3) than (B3) in most applications. Indeed, users of

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[13] The probability measure $\int_{\mathbf{M}} \mu\, dx(\mu) \in \mathbf{M}$ is defined by

$$\left[\int_{\mathbf{M}} \mu\, dx(\mu)\right](S) = \int_{\mathbf{M}} \mu(S)\, dx(\mu), \qquad \text{all } S \in \mathscr{C}.$$

[14] Another key condition, of course, is the "axiom of closure": $\mathrm{Lot}(\mu) \in M$; this, like its analog $\mathrm{Sel}(S) \in A$ in the other theory, is implicit in our definitions.

[15] Marschak [11].

[16] The general question of when a selection space is representable as a *subset* of a Boolean algebra has a certain formal resemblance to the question of when a mixture space is representable as a *subset* of a linear vector space (see [5]), with closure under Boolean addition in the one case corresponding to convexity in the other.

[17] Certain other properties of $u(A)$ can be inferred from (A3); they are not, however, as strong as the convexity of the range $u(M)$ that we get from (B3).

utility theory—economists in particular—are often content to omit risk and probability from their models. On occasion, they will even allow their entrepreneurs to form markets for almost everything under the sun *except* man-made lotteries. This exception is of course tied to a desire to avoid taking the step from ordinal to cardinal utilities.[18] On the other hand, users of utility theory can seldom get very far if they eschew models involving choice situations, or refuse to assign utilities to such situations. For example, it is commonplace to take for granted that the utility to a consumer of a certain income is the supremum of his utilities for the bundles in the budget set defined by that income; this is just an instance of our (A3). Moreover, the case for allowing infinite menus, once finite menus are admitted, is at least as strong as the case for taking the step from finite to infinite lotteries. So boundedness seems forced on us by "Blank Check" as soon as we adopt the real numbers as an ordinal utility scale, and long before gambling and cardinality enter the picture.

In sum, the thrust of these remarks has been toward showing that the unbounded utility issue is independent of the considerations surrounding the St. Petersburg paradox. On the one hand, the Bernoulli game can be modified to be (mathematically) bounded without losing whatever power it might have to make risk-linear money look absurd. On the other hand, the logical challenge that the Bernoulli game raises against unbounded utilities (as opposed to "unreasonably large" but finite utilities) can be mounted equally well in a more basic setting, not involving lotteries at all.

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A Problem with Public Sector Preferences*

In many analyses of the public sector, utility for public sector states is induced on those states from a more primitive utility function defined on some larger space. However, such an induced utility function may not inherit properties of the primitive utility function, such as homotheticity, monotonicity, and separability.[1]

Let $U(z, x)$ be a continuous utility function defined on the product space $Z \times X$, where Z is interpreted as the private (commodity) space and X as the public space. A "natural" definition of an induced utility function on the public space X is

$$V(x) = \max_{z \in B(p,x)} U(z, x),$$

where $B(p, x)$ is a compact (budget) set of available private commodities given prices p and the public state x.[2] Barr and Davis [1], Hinich, Ledyard, and Ordeshook [2], and Zeckhauser and Weinstein [6] have all considered such a function, $V(x)$. A problem is that this function may not be homothetic, monotonic, or separable, even if $U(z, x)$ is.

RESULT 1. If $B(p) = \{(z, x): z \in B(p, x)\}$ is compact, then $V(x)$ is neither monotonic nor homothetic, regardless of whether $U(z, x)$ is homothetic or monotonic.

Proof. Let $(\bar{z}, \bar{x})$ maximize $U(z, x)$ on $B(p)$. Then $\bar{x}$ maximizes $V(x)$ and hence $V(\bar{x}) > V(\bar{x}/2)$ and $V(\bar{x}) > V(2\bar{x})$. These inequalities show that $V(x)$ is neither monotonic nor homothetic. Q.E.D.

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[1] A utility function is *monotonic* if $y > y'$ implies $U(y) > U(y')$, *homothetic* if $U(y) = U(y')$ and $\lambda > 0$ implies $U(\lambda y) = U(\lambda y')$, *separable* if there are functions $V_k(y_k)$ and $F(\cdot)$ such that $U(y) = F(\Sigma V_k(y_k))$ when $y = (y_1, ..., y_n)$ (see [3]).

[2] More generally $V(x)$, with appropriate specifications of z, x, and $B(p, x)$, can be used to analyze other decisions such as all-or-nothing offers, credit or quantity rationing, and possibly strategy choice in game theory.

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It is easily verified that if $B(p)$ is not compact, $V(x)$ may be homothetic or monotonic independently of whether $U(z, x)$ is.

The separability of $V(x)$, without conditions on $B(p, x)$, is independent of the separability of $U(z, x)$. To show this, we present two examples. The first shows that $U(z, x)$ can be separable while $V(x)$ is not and the second that $V(x)$ can be separable while $U(z, x)$ is not.

EXAMPLE 1. Let

$$U(z, x) = \sum_j \alpha_j \log z_j + \sum_i \beta_i \log x_i ,$$

where $\sum_j \alpha_j = 1$, and

$$B(p, x) = \{z : p_z z \leqslant M - t(x)\},$$

where $t(x)$ is any *nontrivial* function of x. From maximizing $U(z, x)$ with respect to z on $B(p, x)$, we get

$$z_j = \alpha_j[M - t(x)]/p_{z_j} ,$$

and thus

$$V(x) = \sum_j \alpha_j \log\{\alpha_j[M - t(x)]/p_{z_j}\} + \sum_i \beta_i \log x_i ,$$

which is not separable, since $\log[\alpha,(M - t(x))/p_{z_j}]$ is not linearly separable in x.

EXAMPLE 2. Let

$$U(z, x) = \prod_j z_j^{\alpha_j} \Big(\sum_i \beta_i \log x_i\Big)\Big(\prod_{i=1}^{3} x_i\Big)^{-1}$$

with $\sum_j \alpha_j = 1$, and

$$B(p, x) = \Big\{z : p_z z \leqslant M\Big(\prod_{i=1}^{3} x_i\Big)\Big\}.$$

Notice $U(z, x)$ is not separable between the x_i's (although it is separable between z and x, and the z_j's) but $V(x)$ is separable in the x_i's, since

$$V(x) = M \prod_j (\alpha_j/p_{z_j})^{\alpha_j} \Big(\sum_i \beta_i \log x_i\Big),$$

after substituting $z_j = \alpha_j M(\prod_{i=1}^{3} x_i)/p_{z_j}$ into $U(z, x)$, as in the first example.

In general then, $V(x)$ may or may not be separable, depending on the particular specification of $U(z, x)$ and $B(p, x)$. If $U(z, x)$ is separable, we have $U(z, x) = F(\sum v_j(z_j) + \sum v_i(x_i))$. Let $\hat{z}(x) = (\hat{z}_1(x), ..., \hat{z}_m(x))$ be defined so that $V(x) = U(\hat{z}(x), x)$.

RESULT 2. Let $U(z, x)$ be separable. Then $V(x)$ is separable if and only if there are functions $w_j^i(x)$ such that $v_j(\hat{z}_j(x)) = \sum_i w_j^i(x_i)$.[3]

The proof is obvious from the definitions, but the result does have some application. If $B(p, x)$ is constant in x, then $\hat{z}(x)$ is a trivial function of x and the result can be used to show separability. In this case, $B(p, x)$ constant in x, an envelope theorem of Rader [5, Theorem 6, p. 919] shows that $V(x)$ inherits most of the properties of $U(z, x)$, for example, concavity and differentiability almost everywhere.

A second application of result 2 is with $U(z, x)$ constant in x. In this case $U(z, x) = \bar{U}(z)$ and we may define $V(x) = \max_{z \in H(x)} \bar{U}(z)$, where $H(x)$ is some set depending on x. There are two interpretations for this formulation, that of utility on trades [4] and, alternatively, indirect utility. With utility on trades, x is interpreted as a trade, and $H(x)$ as the set of feasible final consumptions arising from the trade x. For indirect utility, we interpret x as the price income vector, and $H(x)$ as the budget set.

DISCUSSION

Example 1 is a fairly standard case in which x is interpreted as the allocation of public goods and $t(x)$ is their (*tax*) cost. Given this interpretation, the separability of $V(x)$ should not be expected nor assumed in analyzing the public sector. Notice this is so even though $U(z, x)$ and its associated indirect utility function are separable (if $t(x)$ is linear).

Example 2 is a rather perverse case showing separability of $V(x)$. One interpretation is that x_i, $i = 1, 2, 3$ represents one minus the percentage income tax for say federal, state, and local income taxes. The individual need not be perverse (prefer higher to lower tax rates) for appropriate values of x, β, and α, but it is doubtful whether the example is a "standard" case.

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[3] A related, and stronger, result shows how difficult it is to derive necessary implications of the separability of $V(x)$. Suppose $U(z, x)$ is weakly separable between Z and X; i.e., there exist functions, $u^z(z)$, $u^x(x)$, and $U^*: R^2 \to R$, such that $U(z, x) = U^*(u^z(z), u^x(x))$, and in addition, for each such set of functions suppose that there are functions, $w_i(x_i)$, $i = 1, ..., m$, so that $u^z(\hat{z}(x)) = \sum_i w_i(x_i)$. Then $U(z, x)$ is strongly separable in $Z, X_1, X_2, ..., X_m$, *if and only if $V(x)$ is strongly separable in $X_1, X_2, ..., X_m$.*

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Pitfalls in the Theory of Fairness

1. INTRODUCTION

Egalitarian allocations are in general Pareto- inefficient; likewise, fair (envy-free) allocations that are Pareto efficient need not exist under the standard assumptions on the economic environment. Thus, if one's values conform to either egalitarianism or to freedom of envy an inescapable first-best equity-efficiency trade-off has to be confronted. The concept of egalitarian-equivalent allocations, recently advanced in [7], provides a possible way out of this trade-off for egalitarian oriented societies. This paper is concerned with mitigating the first-best equity-efficiency trade-off for value systems based on the desirability of freedom of envy.

2. THE THEORY OF FAIRNESS

An allocation is said to be fair if no person in the economy prefers anyone else's consumption bundle over his own (see [2]). In other words, a fair allocation is free of envy. The individualistic nature of the fairness idea is attractive and spiritwise consistent with Paretian welfare economics. The appealing features of the fairness criterion from a distributional equity viewpoint are that it treats economic agents symmetrically (in an obvious sense), is ordinal in nature, and is devoid of interpersonal welfare comparisons (since only intrapersonal utility comparisons are involved). It should be mentioned that from the viewpoint of political philosophy it is not entirely clear whether a concept of equity based on envy relationships can be morally acceptable (see [8, pp. 530–514]). Be that as it may, the concept of fairness is certainly interesting enough to warrant investigation of its analytical properties.

From the viewpoint of welfare economics, the major drawback of the fairness criterion lies in its being inconsistent with the Pareto-efficiency principle. Specifically, due to Pazner and Schmeidler [5], it is now known that even under the classical convexity and selfishness assumptions on the economic environment, allocations that are both fair and Pareto efficient will not always exist in economies with production. In light of the general acceptance of the Pareto criterion this presents a fundamental difficulty with the concept of fairness.

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This fact prompted several attempts at modifying the concept of fairness in the hope of arriving at a reasonable equity criterion which would be consistent with Pareto efficiency. I turn now to a critical evaluation of the attempts made so far.

3. EXISTING CONCEPTS OF FAIRNESS

(1) The first concept I wish to discuss is Varian's [10] interesting notion of *wealth-fair* allocations (called fair* in [9]). An allocation is said to be *wealth-fair* if no individual prefers the consumption-output bundle of anyone else over his own. In other words, a wealth-fair allocation is envy free in the sense that no person envies the complete position (defined as including the goods and leisure consumed *and* output produced) of any other person. As shown in [9], wealth-fair allocations are consistent with Pareto efficiency under the standard assumptions on the economic environment.

The problem of course is that unless the production technology is additively separable in each agent's labor time inputs, it is impossible to impute output according to individual productivity. In other words, production processes are more often than not of such a joint nature as to make it impossible to disentangle individual contributions to total (observed) output. But in linear production economies, for instance, the concept is well defined and the efficiency result of interest.

Confining our attention to those very special cases in which the wealth-fairness concept is well defined, it becomes of interst to discuss its normative significance. As noted by Varian himself [10], if one agent cannot possibly produce what another agent produces then no envy complaint can ever be raised by the first agent against the second. This example illustrates nicely the implicit sanctification of productivity (a morally irrelevant characteristic in itself)[1] underlying the wealth-fairness criterion. In other words, it is first and foremost a "to each according to productivity" kind of slogan. It somewhat mitigates this near-libertarian precept by requiring that those consumption commodities that are not resultant upon production should (loosely speaking) be divided in a fair manner (according to the original definition of fairness). In any case, ceteris paribus, this criterion penalizes the unable (in the productive sense). Horizontal equity (Musgrave's [3] suggestive term for the equal treatment of equals) is taken to mean under it that only persons who are identical both in preferences and in productivity will be treated equally. Persons who are identical only in preferences on the other hand, will be treated differentially to an extent entirely dependent upon their differential productivities. If it is agreed that the subject matter of economic equity relates to preferences only (i.e., if it is agreed that the proper axiom of horizontal equity states that people with identical preferences ought to enjoy the same welfare level in the sense of being assigned a bundle lying on the same indifference surface)[2] then the wealth-fairness concept ought to be rejected on ethical grounds. Also, it would seem that wealth-fair allocations can under no circumstances be rationalized in terms of hypothetical contractual agreements in Rawls' [8] original position.

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[1] One may, or perhaps even should, distinguish here between differential productivites that are due to differential "innate abilities" (or, more generally, differential productivities that are essentially resultant upon a random act of nature) and those that are attributable to self-chosen actions by individuals (schooling, self-study, specialization, etc.). For the second category, under (hard to define) conditions of equal opportunity, the wealth-fairness concept has a certain normative significance. But I fail to see any possible moral justification for this concept in the case of innate (or, more generally, exogenous) productivity differentials.

(2) The second concept of interest is that of *income-fairness* suggested by Pazner and Schmeidler [6] and further discussed by Varian [9, 10]. An allocation is said to be *income-fair* if at the efficiency prices supporting this allocation the value of each person's consumption-cum-leisure bundle is equal. In other words, an income-fair allocation calls for the perfect equalization of potential income (sometimes called implicit income).

Unlike the wealth-fairness concept, under which each person has a full ownership right over his "natural" endowment of time, under the income-fairness criterion each person is thus effectively assigned an equal property right (or share) in everybody's endowment of time (including his own). The distribution of skills is thus viewed as a common pool of productive resources to be shared, in a sense, equally among all members of the society. Unlike the purely private good nature of individual productive skills implied by the wealth–fairness criterion (under which redistribution of the fruits of labor is prohibited), the income-fairness criterion is thus consistent with the general Rawlsian viewpoint.

As shown in [6], under the standard assumptions on the economic environment the income-fairness criterion is always consistent with Pareto efficiency, a desirable property. There is, however, one major drawback with the income-fairnesss concept. It does not satisfy horizontal equity as defined above, i.e., at an income-fair allocation two persons with identical preferences will not, in general, be assigned bundles that lie on the same indifference surface whenever their productivities differ (see [6]; for a clear illustration of this fact see [4, footnote 11, pp. 630–631]). In general, the more able a person, ceteris paribus, the more penalized he is relative to an unable one. Income-fair allocations therefore discriminate among people in a manner that is diametrically opposed to the discrimination taking place at wealth-fair allocations. In any event, as I consider the property of horizontal equity to be of importance, its violation under the income-fairness criterion is troublesome in my opinion.

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[2] It might be appropriate at this point to correct a minor error by Varian [10, p. 246], who says that when all agents have the same tastes but different abilities, an equal division of goods and labor leads to a fair and Pareto-efficient allocation. Actually, the only inference that can be made in this instance is that at an allocation that is both fair and Pareto-efficient (such indeed exists in this case and is unique under the standard assumptions) the individuals get consumption–leisure bundles that all lie on the same indifference surface. Since abilities differ, the equal division is generally not Pareto- efficient (due to the leisure margin).

(3) The last existing concept of fairness is that recently suggested by Daniel, who offers the following definitions. If the number of people who envy a person is equal to the number of people that he envies, then he will be said to be *balanced* with respect to envy at that allocation. An allocation is said to be *balanced* if everyone is balanced at it. He then goes on to prove that under the standard assumptions on the economic environment (to which a so-called nondegeneracy assumption is added) there always exist Pareto-efficient allocations that are balanced (see [1], some misattributions in which are rectified in [11]).

Thus the balancedness criterion of fairness is consistent with the Pareto-efficiency requirement, a result of some interest in itself. The question, however, is how normatively appealing is the balancedness requirement in the first place. Consider the extreme case of a large society in which every individual envies every other at some (Pareto-efficient) allocation. This allocation clearly satisfies the balancedness requirement. Yet, in such a large society what every individual can at most know is that he envies every other individual. There is really no way (if we want to be reasonable about it) for him to know that every other individual also envies him unless he is told so (say by an "ethical observer"). Thus the very spirit of the fairness idea (according to which each person can evaluate his position relative to others without any outside help) is lost. Furthermore, it is dubious whether the individual would feel much better even if he were told about this mysterious property of universal mutual envy. True it is a phenomenon that treats all agents symmetrically in a sense; but, it would seem to me that it would very likely give to the no less symmetrical situation in which everybody is at everybody else's throat. Hardly a stable social situation, and certainly not one in agreement with our intuitive vision of the good society. In brief, it is hard for me to take seriously the balancedness requirement as a plausible distributional equity criterion.

4. New Concepts of Fairness

In the light of the above discussion, the question arises whether one could advance new concepts of fairness which are free of the difficulties associated with earlier attempts. I shall now present two new concepts, the second of which I believe to deserve special attention.

(1) In the light of the tremendous informational requirements in large societies placed upon any individual who has to perform the intrapersonal envy comparisons called for by all the previous fairness criteria,[3] one could think it operationally meaningful to advance the following simplified fairness test. Let each person at any allocation determine whether he is better off with his bundle as compared to the average bundle in the economy. The informational requirements here are minimal in the sense that all that each agent has to know is his own bundle, the aggregate bundle,[4] and the number of agents in the economy. An allocation at which no individual prefers the average bundle over his own will be said to be *per-capita-fair*. In terms of the information likely to be (costlessly) available to any individual, this fairness requirement is very sensible.

Do per-capita-fair allocations exist? Of course. Simply give to each person an identical (egalitarian) consumption-cum-leisure bundle.[5] Will this egalitarian bundle be Pareto efficient? Generally not. In fact, the major difficulty with the concept of per-capita-fair allocations is that, like the original fairness criterion, it is inconsistent with the Pareto-efficiency requirement. To see that, simply observe that in any standard two-person production economy in which no fair and Pareto-efficient allocation exists, there can exist no per-capita-fair and Pareto-efficient allocation. For, under convex preferences, the fact that at every Pareto-efficient allocation in such an economy at least one of the individuals envies the other implies that the average bundle is also preferred to the bundle that he has. So no Pareto-efficient and per-capita-fair allocation can exist in such an economy and this simple counterexample is enough to rule out the interesting notion of per-capita-fairness on grounds of inconsistency with the Pareto principle.

---

[3] The original fairness criterion requires for its application that each person knows at any allocation the consumption–leisure bundle that every other consumer gets. The wealth-fairness criterion requires in addition that each person know every other person's productivity (when the latter is well defined)! The income-fairness criterion requires each person to know every other person's actual ("market") income, consumption of leisure, and efficiency wage. The balancedness criterion requires each person to know every other person's consumption–leisure bundle and the preferences of every other person! Of all these, the income-fairness criterion is the least demanding from the informational viewpoint, but even it is extremely informationally demanding in a large society.

[4] Including of course the aggregate number of hours worked (and hence the aggregate amount of leisure). This datum is available in virtually every country.

[5] Note that in a production economy assigning to everybody the average bundle may not be feasible (as for instance will be the case when the "able" work more hours than the "unable"). But in order to apply the per-capita-fairness criterion, the feasibility problem is of no concern; each person asks himself whether he is better off with his bundle as compared to the (possibly infeasible) average bundle. Observe also that the same feasibility problem already arises with the original fairness criterion.

(2) The second novel fairness concept which comes to mind is motivated by the concept of *egalitarian-equivalence* recently introduced by Pazner and Schmeidler [7]. An allocation will be said to be *fair-equivalent* if there exists a fair allocation in some hypothetical economy in which each person enjoys the same welfare level as that enjoyed by him at the allocation under consideration. In other words, an allocation is *fair-equivalent* if and only if its underlying welfare distribution could have been generated by a fair allocation in some hypothetical economy.[6] If such an allocation exists and if our conception of the "good society" is that of an envy-free society, the normative significance of any fair-equivalent allocation derives from the fact that each person is indifferent between living in the actual economy (at the said allocation) and living in the "good" (envy-free) reference economy.

Regarding the question of whether or not the fairness-equivalence criterion is consistent with the Pareto-efficiency principle, recall first the following definition. An allocation is said to be *egalitarian-equivalent* if its underlying welfare distribution could have been generated by an egalitarian economy. Clearly, any egalitarian-equivalent allocation is also fair-equivalent. As shown in [7], Pareto-efficient and egalitarian-equivalent allocations always exist under (even weaker than) the standard assumptions on either exchange or production economies. This establishes the consistency of the fairness-equivalence requirement with the Pareto-efficiency criterion, an important property. No less important, horizontal equity is also satisfied.

It clearly is the case that the set of fair-equivalent allocations will always contain the set of egalitarian-equivalent allocations. Therefore, the restriction imposed on the Pareto set by the egalitarian-equivalence criterion is more discriminating (i.e., rule out more Pareto-efficient allocations as being normatively admissible on equity grounds). This might be considered by some as a distinct advantage of this criterion. But if one's view of the good society is that of freedom of envy, the fairness-equivalence criterion will seem more basic. And since those who are egalitarian-minded can always restrict their attention to the egalitarian-equivalent subset of the set of fair-equivalent allocations, I think it wisest to let the reader decide for himself which of these two concepts (if any) better conforms to his own values. The fact that they both satisfy the horizontal equity requirement is in my mind an important property when comparing either of them with the modified fairness criteria discussed in Section 3.

Like the modified fairness criteria discussed in Section 3, the two equivalence criteria are however much more informationally demanding than the original fairness criterion. The appealing simplicity with which each individual could perform his envy comparisons under the original notion of fairness is thus lost. In this sense, it would seem that the original theory of fairness may have reached a dead end. From the viewpoint of the omniscient-planner approach to Paretian welfare economics, though, the two equivalence criteria seem to offer a promising avenue out of what I like to think of as being the disturbing problem of the *first-best equity-efficiency trade-off.*

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[6] Hypothetical because, again, feasibility (in the actual economy) of the fair allocation (in the reference economy) is not guaranteed. What is guaranteed, however, is that the underlying welfare distribution is feasible in the actual economy.

MATHEMATICAL APPENDIX TO SECTION 4

Since the concepts and propositions of this section are new, the following mathematical treatment intends to make precise the ideas and statements presented in the text. The notation is consistent with that in [7].

Let T denote the finite set of economic agents and let R^l_+ denote the nonnegative orthant of the Euclidean space of dimension l, the set of consumption commodity bundles. Each t in T has a preference relation $\gtrsim_t$ on R^l_+ which is assumed to be *strongly connected, transitive, continuous, monotonic, and convex* (i.e., for all x, y, z in R^l_+, the following hold: $x \gtrsim_t y$ or $y \gtrsim_t x$; $x \gtrsim_t y$ and $y \gtrsim_t z$ imply $x \gtrsim_t z$; the sets $\{x' \in R^l_+ \mid x' \gtrsim_t x\}$ and $\{x' \in R^l_+ \mid x \gtrsim_t x'\}$ are closed in R^l_+; $x \succ_t y$ implies $\lambda x + (1 - \lambda)y \succ_t y$ for $0 < \lambda < 1$; and $x > y$ implies $x \succ_t y$, where inequalities between vectors in R^l_+ hold coordinatewise by definition, and the relations $\succ_t$ and $\sim_t$ are induced by $\gtrsim_t$ in the usual way).

The set of technologically feasible production plans is denoted by K $(K \subset R^l)$. As usual, if $z \in K$ then the negative coordinates of z denote inputs and the positive coordinates of z denote outputs. Let $W = \{\omega + K\} \cap R^l_+$, where $\omega \in R^l_+$ is an aggregate initial commodity vector, denote the feasible aggregate final consumption vectors. It is assumed that W is a compact and convex set with nonempty interior. Free disposal is also assumed, i.e., $x \leq y \in W$ implies $x \in W$.

The *economy* is formally defined as the vector $(T, R^l_+, \{\gtrsim_t\}_{t \in T}, W)$. An *allocation* is a T list of elements of R^l_+ whose sum belongs to W. An *assignment* is a T list of elements of R^l_+ (whose sum does not necessarily belong to W). In other words, while an allocation is a feasible T list of consumption (cum-leisure) bundles, an assignement is a T list of such bundles which is not necessarily feasible. An assignement is denoted by $\{z_t\}_{t \in T}$ or simply $\{z_t\}$. An allocation $\{x_t\}$ is *Pareto efficient* if for any other allocation $\{y_t\}$ the implication $(\forall t \in T, y_t \gtrsim_t x_t) \Rightarrow (\forall t \in T, y_t \sim_t x_t)$ holds. An allocation $\{x_t\}$ is *fair* if for all t and t' in T: $x_t \gtrsim_t x_{t'}$.

DEFINITION 1. An allocation $\{x_t\}$ is said to be *per-capita-fair* if for all t in T: $x_t \gtrsim_t \bar{x}$, where $\bar{x} = (\sum_{t \in T} x_t / |T|)$.

PROPOSITION 1. *There exist economies as defined above in which no Pareto-efficient allocation is per-capita-fair.*

Proof. By counterexample. Consider a two-person economy as in [5], where no Pareto-efficient allocation is fair. At any Pareto-efficient allocation $\{x_1, x_2\}$ in such an economy, either $x_2 \succ_1 x_1$ and/or $x_1 \succ_2 x_2$. Suppose without loss of generality that $x_2 \succ_1 x_1$. By the convexity of preferences, $\bar{x} = (x_1 + x_2)/2 \succ_1 x_1$. Q.E.D.

DEFINITION 2. An allocation $\{x_t\}$ is said to be *fair-equivalent* if there exists an assignment $\{z_t\}$ such that for all t and t' in T: $x_t \sim_t z_t$ and $z_t \succsim_t z_{t'}$.

Thus a fair-equivalent allocation $\{x_t\}$ is agentwise indifferent to a fair allocation in an economy in which the assignment $\{z_t\}$ is feasible.

PROPOSITION 2. *In an economy as defined above there always exist Pareto-efficient allocations that are fair-equivalent.*

For the proof of Proposition 2 we need a definition and a lemma.

DEFINITION 3. An allocation $\{x_t\}$ is said to be *egalitarian-equivalent* if there is a bundle z in R^l_+ such that for all t in T: $x_t \sim_t z$.

LEMMA. *If an allocation $\{x_t\}$ is egalitarian-equivalent then it is fair-equivalent.*

Proof. By the definition of an egalitarian-equivalent allocation, there exists a constant assignment $\{z\}_{t \in T}$ such that $x_t \sim_t z$, for all t in T. Since $z \succsim_t z$ for all t in T, $\{x_t\}$ is fair-equivalent. Q.E.D.

Proof of Proposition 2. As shown in [7], in an economy as defined above there always exist Pareto-efficient allocations that are egalitarian-equivalent. By the Lemma, Proposition 2 then follows. Q.E.D.

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A General Equilibrium Framework for the Divvy Economy*

1. INTRODUCTION

Dantzig, in [2], formulates a model for resource allocation in the so-called "Divvy Institutionalized Economy." The author proves the existence of a general equilibrium solution to the economic problem (in terms of quantities and prices of input factors and final goods) which at the same time satisfies agreed upon shares of monetary flows allocated to the groups controlling the input factors, and to the groups that demand the final goods for consumption. The agreement on the shares is carried out by a political process, while market mechanisms adjust the prices of primary resource inputs and the relative sizes of the consumer groups until those shares are satisfied. The relationship between inputs and outputs or transformation technology is presented in an input–output format.

The major innovation of the model is the integration of institutional forces together with market mechanisms to explain the resource allocation problem. Social phenomena like collective bargaining, congressional budget approval, propositions to index the prices of raw materials to the prices of certain final goods, which are difficult to explain using the classic economic framework of analysis, will be better understood under the Divvy formulation.

The purpose of this paper is to study the Divvy Economy in terms of neoclassical economics and derive some implications for the traditional theory. Our focus is then behavioral; i.e., we maintain the models of consumer utility maximization and producer efficiency maximization. A summary of the model that is going to be the background of our exposition is presented in the Appendix.

We will assume 1,..., r resource groups and 1,..., s consumer groups. There are 1,..., n economic sectors which buy resources x_i, $i = 1,..., r$ to produce final goods y_k, $k = 1,..., n$. p_k, $k = 1,..., n$ will represent prices of final goods and λ_i, $i = 1,..., r$ prices of primary resources; the relative sizes of the groups will be given by μ_j, $j = 1,..., s$.

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2. A Welfare Maximization Problem Compatible with the Divvy Economy

Consider the following optimization problem

$$\max_{\mu,x,y\geq 0} U = \prod_{j=1}^{s}\prod_{k=1}^{n}(y_{jk}\mu_j)^{\beta_{jk}} = \prod_{j=1}^{s}\left(\prod_{k=1}^{n} y_{jk}^{\beta_{jk}}\right)\mu_j^{\sum_k\beta_{jk}}$$

s.t.

$$\sum_{j=1}^{s} y_{jk}\mu_j \leq q_k, \qquad \forall k, \tag{2.1.1}$$

$$q_k = \prod_{i=1}^{r} x_{ik}^{\alpha_{ik}}, \qquad \forall k, \tag{2.1.2}$$

$$\sum_{k=1}^{n} x_{ik} \leq h_i, \qquad \forall i. \tag{2.1.3}$$

The objective function is a welfare function for the society, where y_{jk} is the quantity of final good k consumed by a member of group j, and μ_j is its relative size. Equation (2.1.1) is an availability constraint for each final good, and (2.1.2) represents the technology producing k. Equation (2.1.3) is an availability constraint in primary inputs. Problem (2.1) does not differ significantly from the classical welfare maximization problems [3]. The only innovation is to consider the groups rather than single individuals in the index of welfare that has been chosen. If p_k is the multiplier in the combined constraints (2.1.1) and (2.1.2) and λ_i is the multiplier in (2.1.3), we can write the two behavioral models associated with (2.1).

The consumer utility maximization sub model would be

$$\max_{y\geq 0} U = \prod_{j=1}^{s}\left(\prod_{k=1}^{n} y_{jk}^{\beta_{jk}}\right)\mu_j^{\sum_k\beta_{jk}} \tag{2.2}$$

s.t.

$$\sum_k\sum_j p_k y_{jk}\mu_j = 1,$$

where 1 is a normalized value for the income of the society, so now we will also have normalized prices. In the Divvy Economy the y_{jk} are considered given as somehow "typical" consumption patterns for the groups, and in that case the optimization is carried over μ_j. Equation (2.2) converts then to

$$\max_{\mu\geq 0} U = k_1\prod_{j=1}^{s}\mu_j^{\beta_j} \tag{2.3}$$

s.t.

$$\sum_{k}\sum_{j} p_k y_{jk} \mu_j = 1, \tag{2.3.1}$$

where $\beta_j = \sum_{k=1}^{n} \beta_{kj}$, $\beta_j \geqslant 0$, $\sum_{j=1}^{s} \beta_j = 1$, $k_1 = \prod_{j=1}^{s} \prod_{k=1}^{n} y_{jk}^{\beta_{jk}}$. If v is the Lagrange multiplier of (2.3.1), the necessary conditions give, after solving for $v = k_1 \prod_{j=1}^{s} \mu_j^{\beta_j}$,

$$\sum_{k=1}^{n} p_k y_{jk} \mu_j = \beta_j \qquad \forall j, \tag{2.4}$$

together with $\mu_j \geqslant 0$ and (2.3.1).

Similarly, the producer efficiency maximization sub model would be,

$$\min_{x \geqslant 0} \sum_{i=1}^{r} \lambda_i x_i \tag{2.5}$$

s.t.

$$y \leqslant \prod_{i=1}^{r} x_i^{\alpha_i}. \tag{2.5.1}$$

Equation (2.5) is actually an aggregated model of the index of output across sectors; i.e., (2.5.1) would actually be written as

$$y = \prod_{k=1}^{n} y_k^{\beta_k} = \prod_{k=1}^{n} \prod_{i=1}^{r} x_{ik}^{\alpha_{ik}} = \prod_{i=1}^{r} x_i^{\alpha_i},$$

with $\alpha_{ik} = \beta_k \alpha_i$ and $\sum_{i=1}^{n} \alpha_i = 1$, $\alpha_i \geqslant 0$.

Since the Divvy Economy takes λ as the variables, we want the equivalent problem to (2.5) in terms of λ. This is provided by the "dual" problem of (2.5). The concept of duality and the specific dual for problems of the form (2.5) are treated in [4].

$$\min_{\lambda \geqslant 0} \sum_{i} \lambda_i x_i \tag{2.5$'$}$$

s.t.

$$k_2 \prod_{i=1}^{r} \lambda_i^{\alpha_i} \geqslant 1, \tag{2.5.1$'$}$$

where $k_2 = \prod_{i=1}^{r} \alpha_i^{-\alpha_i} y$, and $\sum_i \lambda_i x_i{}^* = k_2 \prod_{i=1}^{r} \lambda_i^{\alpha_i}$, for $x_i{}^*$ solving (2.5).

The first-order condition to (2.5)′ gives

$$\lambda_i x_i \Big/ \left(\sum_{i=1}^{r} \lambda_i x_i \right) = \lambda_i x_i = \alpha_i \qquad \forall i, \tag{2.6}$$

or in more disaggregated form,

$$\lambda_i x_i = \sum_{k=1}^{n} \lambda_i x_{ik}, \qquad \sum_{k=1}^{n} x_{ik} = x_i \qquad \forall i. \tag{2.7}$$

Equations (2.4) and (2.7), together with the accounting conditions that industry makes zero profit, $\sum_{i=1}^{r} \lambda_i x_{ik} = \sum_{j=1}^{s} p_k y_{jk} \mu_j$ and aggregate value of inputs equal aggregate value of outputs, characterize the general equilibrium conditions. Since p_k, y_{jk}, and x_{ik} to achieve y_{jk} are assumed given, the conditions are stated in terms of λ and μ.

This can be easily seen by assuming a linear aggregation formulation. Let $L(n \times n)$ be an input–output matrix of interindustry transactions, $R(r \times n)$ a matrix of resource inputs per unit of output.

The zero profit conditions give $p = \lambda R L^{-1}$, assuming $|L| \neq 0$. In the real sector, *the relation between inputs and final demand can be written as* $x = RL^{-1}y$. Now we can define exactly $[x_{ij}] = [R_i L^{-1} y_j]$, where x_{ij} is the quantity of input factor i per unit of output j.

Using these results, (2.4) and (2.7) would be now

$$\sum_{i=1}^{r} \lambda_i x_{ij} \mu_j = \beta_j \qquad \forall j, \tag{2.8}$$

$$\sum_{j=1}^{s} \lambda_i x_{ij} \mu_j = \alpha_i \qquad \forall i. \tag{2.9}$$

Equations (2.8) and (2.9) are equivalent to (18) in [2, p. 380] for $\beta_j = \delta_j$, $\alpha_i = \gamma_i$, and $M_{ij} = x_{ij}$.

3. CRITIQUE OF THE MODEL

In the previous section we have shown that the behavioral models of consumer utility maximization and producer efficiency maximization in the way they are formulated in classical economic theory, under assumptions similar to those in the Divvy Economy (fixed consumption pattern for each group), give results that are consistent with the ones implied by the Divvy Economy, in terms of solutions satisfying general equilibrium conditions. However, the behavioral assumptions of both models are very different and would call for different understandings of the economic problem. The decision on which model truly represents the actual behavior is difficult to make because the shares of flows would have to be observed from empirical results, and either model could claim that they were generated under its assumptions.

But apart from this ambiguity, the models deserve other comments on their assumptions. The Divvy Economy, we think, makes a valuable and justifiable point when arguing that the shares of flows are affected by the political process, something which in the classical economic result is not very often considered.[1] Nothing is said however, about the dynamics of the political process and what social pressures actually determine those shares. Rather, by introducing a new variable, the adjustable size of the groups, the system has enough degrees of freedom to minimize the effects of political decisions in the economic sector, together with the fact that the shares are stated in monetary terms and not in real ones.

Observing the functioning of the political process, one can claim that there exist social groups bargaining for the shares of the outcome of the economy, in real terms. The groups are rather fixed and very often have strict control of membership to maintain their competitive advantage. To distinguish between resource groups and consumer groups is difficult since, except for retired people, the rest participate directly in the production process (in a broad sense), and a good part of their share is determined already by the remunerations for their contribution. The groups that do not feel satisfied by the strict-economic share that marginal productivity criteria would assign to them, make use of the political system and force redistributive actions by the government. An example would be the "income policy" whose aim is to achieve income redistribution through taxation. Another example at the international level would be the aid programs of the developed countries in favor of the less developed ones, while a manifestation of the power of the resource groups is the ability of the OPEC countries to control the price of oil. In this last case note, however, the concern of those countries in changing the prices so that their share of flows is always maintained in real terms.

The previous analysis suggests new formulations of the resource allocation problem. Although technological and economic relationships must not be put aside, the political power of the social groups in controlling the economic resources and in the decisions about the selections of a particular technological option need to be better understood. The Divvy Economy provides an encouraging starting point.

APPENDIX: FORMULATION OF A GENERAL EQUILIBRIUM PROBLEM CONSISTENT WITH OUR BEHAVIORAL MODELS

Given a vector of final goods y produced by a technological relationship of the form $y = F(x)$, where x is a vector of inputs available in quantities h, for a utility function on y, $U(y)$, the problem is

---

[1] The introduction of budgetary constraints for the government sector in some models [1] could be interpreted as an example of this consideration.

$$\max_y U(y)$$

s.t.

$$y \leq F(x), \qquad x \leq h. \tag{A.1}$$

The solution to (A.1) will satisfy the Pareto condition that $U(y^*) \geqslant U(y)$ for all feasible y.

If p and λ are the vector dual variables associated with the first and second constraint in (A.1), the consumer utility maximization problem subject to the budget constraint

$$\max_y U(y)$$

s.t.

$$py \leq M \tag{A.2}$$

and the producer efficiency maximization (cost minimization) problem

$$\min_x \lambda x$$

s.t.

$$y \leq F(x) \tag{A.3}$$

will provide general equilibrium conditions consistent with (A.1), and consequently satisfying the Paretian condition. Problems (A.2) and (A.3) are called for this reason Paretian rules [3].

The first-order conditions are

$$(\partial U/\partial y) - vp = 0, \tag{A.2.1}$$

$$\lambda - p(\partial F/\partial x) = 0, \tag{A.3.1}$$

where v is the marginal utility of income or scaling factor that converts monetary output of the society into welfare measures in utils. $F(x)$ is assumed homogeneous of degree one, which means that at optimal $py = \lambda x = M$. A general equilibrium solution is the vectors y, x, p, λ satisfying (A.2.1) and (A.3.1), together with $py = M$ and $y = F(x)$.

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